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ELEMENTARY MECHANICS.

INTRODUCTION.

Mechanical Science is that in which the laws of forces, and the effects they produce on bodies, are investigated.

It is subdivided into four Sciences—Statics, Dynamics, Hydrostatics, and Hydrodynamics.

In Statics the effects of forces on solid bodies at rest are examined.

In Dynamics the effects when motion is produced.

In Hydrostatics the effects of forces on fluid bodies at rest are considered.

In Hydrodynamics the effects on fluid bodies when motion ensues.

In the present treatise, Elementary Statics and Dynamics only, are treated of.

The *mass* of a body is the quantity of matter which it contains; and *matter* is defined to be, whatever possesses bulk and affords resistance to the occupation of the same portion of space by other matter.

We are ignorant of the ultimate nature of matter, but we know that dense matter consists of atoms,[*] which have each their peculiar masses constant and unchangeable by any mechanical or chemical means within our reach.

---

[*] See Dr. Daubeny on the Atomic Theory.

The term *subtile matter* has been applied to the agents which cause the phenomena of electricity, heat, &c.

Though evidently closely connected with the development of forces, we as yet only know some of the properties and laws of the effects of these agents upon dense matter. Whenever the term *matter* is used in Mechanics, it is understood to mean what is called above dense matter. The quantity of matter in a body is measured by its inaptitude to receive motion (*inertia*) when acted on by a given force; and is proportional to the weight at the same place on the Earth's surface. So that a body of two, three, &c. pounds weight contains twice, thrice, &c. respectively the matter that a body of one pound does.

We define *force* to be, whatever causes or tends to cause motion, or change of motion in bodies. We see force acting continually around us, and developed by various means, though we cannot trace it to its ultimate origin. We measure forces by their effects, and in *Statics* they are often called *pressures;* being compared with the pressures produced by known weights, they can thus be expressed numerically. We speak, for instance, of a pressure of twelve ounces, of thirty pounds, of two tons, &c. &c. when the unit of measure is an ounce, a pound, or a ton respectively.

In *Dynamics*, forces are measured in two different manners, according to the nature of the problem,—namely, in some cases by the velocity generated in a unit of time; and in other cases by the momentum (or velocity multiplied into the mass of the body moved) generated in a unit of time.

In *Statics*, we continually represent forces by lines of definite lengths. A unit of length being taken to represent the unit of pressure, the length of the line represents the magnitude of the force; its direction represents the direction of the force; and the commencement or first extreme of the line, the point at which the force acts, or its point of application. Lines which are parallel are said to have the same direction.

In order that a statical force may be known, its *magnitude*, *direction*, and *point of application*, must be given.

The weight of bodies, being their gravitation vertically downwards, arises from the attraction of the Earth upon them, according to the laws of universal gravitation. These laws we shall have to consider in the science of Dynamics.

In statical problems we have frequently other forces arising from the effects of the original forces, which have to be considered in the same manner; as the *tension* in cords, and the *reaction* in rods. Unless the contrary is stated, the cord is supposed without weight, perfectly flexible, and to pass perfectly freely round any object which changes its direction: the force applied at one extremity must then be transmitted without loss along the whole length of the cord, or the *tension* in the cord is the same in every part. Unless the contrary is stated, the rods are supposed to be inflexible and without weight; so that a straight rod transmits a force applied at one extremity in the direction of its length, to the other extremity unchanged; thus the *reaction* of the rod equals the original force, and may be supposed to act at any point in it.

In other statical problems there arise forces of a different nature to the original forces, and which therefore have to be considered differently; as the *friction* which arises from the roughness of the surfaces of bodies in contact, and the *adhesion* which arises when one of the surfaces at least is of an adhesive nature. The laws of *friction* have been ascertained, and will be treated of in a distinct chapter. The properties of *adhesive* surfaces are of less theoretical importance, and are seldom discussed in treatises on Mechanics.

By a *particle* of a body, we mean a portion of it whose dimensions are smaller than any possible means of measurement.

By a *rigid* body, we mean one in which the relative positions of its particles remain unchanged.

In some statical problems the properties of flexible and elastic bodies have to be considered. In these the relative positions of the particles change by the action of the forces.

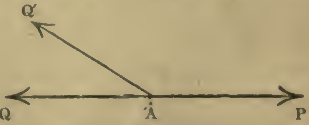
STATICS.

CHAPTER I.

ON FORCES WHICH ACT AT THE SAME POINT.

1. *A force acting at any point is balanced by an equal force acting at the same point in an opposite direction.* It is clear that this must be true; for whatever tendency to motion the point might receive from one of the forces, it would receive an equal and opposite tendency from the other; and these would neutralise each other.

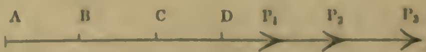
2. *If two forces be in equilibrium at a point, they must be equal in magnitude and opposite in direction.* If possible, let the forces P and Q', acting in the directions of the arrows, as in the figure, keep the point A at rest. Let Q be a force equal and opposite to P; P and Q will balance, and therefore Q produces the same tendency to motion that Q', a different force does, which is absurd.



Definition. The *resultant* of two or more forces is the single force which produces the same mechanical effect as the forces themselves; which are called the *component* forces.

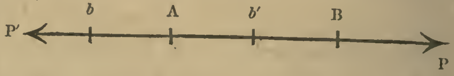
3. *When any number of forces act at a point in the same straight line, the resultant equals the algebraic sum of the component forces.*

First, let all the forces act in the same direction, as in the figure. Let the line AB represent the force P_1. If the point B be rigidly connected with A, we may suppose the force P_2 to act at B, and it will produce the same effect as if it acted directly at A. Let



BC represent the magnitude of P_2. The line AC represents a force equivalent to P_1 and P_2 acting at A; or, their resultant equals their sum. If we had a third force, P_3, represented by CD, we should have the resultant of P_1, P_2, and P_3, a force represented by AD, their sum as before; and so onwards for any number of forces.

When some of the forces act in the contrary direction, we must take the lines which represent their magnitudes on the contrary side of A, and subtract them from the sum of the other forces. Let P be represented by the line AB, and P', a force acting in the opposite direction, by Ab; we may suppose P' to act at B, and measuring Bb' equal to Ab, we have Ab', a line representing the resultant of P and P'. So that, having taken the sum of the forces acting in one direction and the sum of those acting in the contrary direction, the difference of these sums is the resultant force, and it acts in the direction of the forces which form the larger sum. This is equivalent to calling the forces in one direction *positive*, and those in the contrary direction *negative*; and then their algebraic sum gives the magnitude of the resultant, and its sign determines its direction. When the resultant $= 0$ the forces balance, and the condition of equilibrium of forces acting in the same straight line is, that their algebraic sum $= 0$.

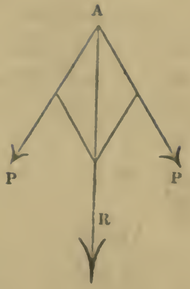


Hence, we may add equal and opposite forces at any point without affecting a system of statical forces: this is called the *superposition* of equilibrium. In the same way we may remove from any point in a system those forces which are equal and opposite. In finding the resultant of given forces, we are said *to compound* them; and when we find the components of a given force, we are said *to resolve* it into its components.

THE PARALLELOGRAM OF FORCES.

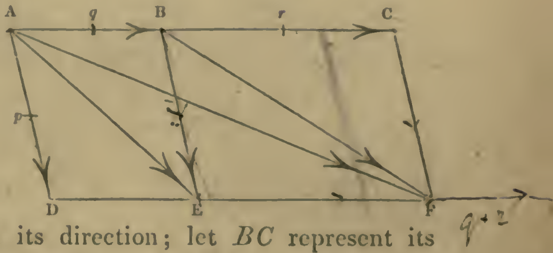
4. Prop. *When two forces act at a point, if the parallelogram upon the lines representing them be completed, the diagonal from that point represents their resultant in magnitude and direction.*

First, to prove that the diagonal is *the direction* of the resultant. If the forces are equal, as P, P in the figure, this is evidently true, for no reason can be assigned why it should incline more to one force than the other, and the diagonal bisects the angle A in which the forces meet.



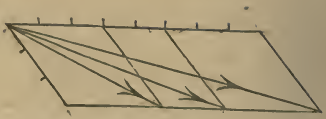
Let us assume that this holds good for forces p and q, and p and r, we shew that it must also be true for a force p and a force $q+r$.

Let p and q act at A, as in the figure, in the directions AD and AB, and be represented in magnitude by these lines respectively. We may consider the force r to be applied at B, a point in its direction; let BC represent its magnitude. Complete the parallelogram $ADFC$, and draw BE parallel to AD. Draw the diagonals AE, AF, BF.

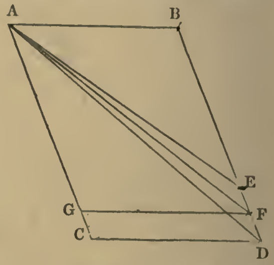


Then, p at A in AD, and q at A in AB, are equivalent to a force in AE which we may suppose to act at the point E. Resolving it again, parallel to the original directions, we have a force p acting at E, in the direction BE, and a force q acting at E in EF. We may suppose them to act at the points B and F in their directions; but p at B in BE, and r at B in BC, are equivalent to a force in BF. Therefore we may suppose all the forces to act at the point F, parallel to their original directions, and F must be a point in their resultant; or force p at A in AD, and forces q and r in AC, have their resultant in the direction of the diagonal AF.

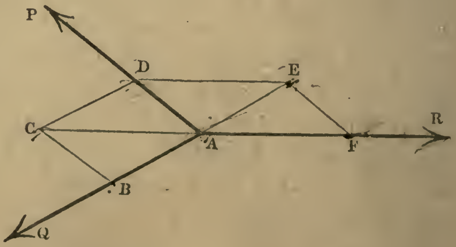
Now, our assumption is true when q and r are each equal to p, therefore the proposition for the direction of the resultant is true for forces p and $2p$: again, putting $q=2p$, $r=p$, it is true for forces p and $3p$, and so for p and np; also, putting np for p, and $q=p$, $r=p$, it is true for np and $2p$, and so onwards



for all commensurable forces. We see also in the annexed figure, that the resultant lies nearer to the greater than the weaker force; and if one of the two forces be increased, the resultant lies still nearer to the increased force.



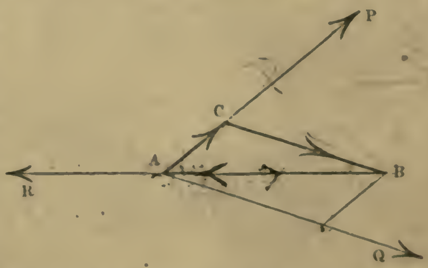
When the forces are incommensurable, the proposition still holds good. Let the lines *AB*, *AC* represent the incommensurable forces: complete the parallelogram, and draw the diagonal *AD;* then *AD* represents the direction of the resultant. If not, let some other line, as *AE*, be its direction. Take a quantity which divides *AB* without remainder, and being applied to *AC*, leaves a remainder *GC* less than *ED*. Complete the parallelogram *ABFG*, and draw the diagonal *AF*. Now *AB*, *AG*, represent commensurable forces, and therefore their resultant is in the direction *AF;* but *AE*, the resultant of *AB* and greater force *AC*, is nearer *AB* than the resultant of *AB* and the less force *AG*, which is impossible. Similarly it can be proved, that no other direction than *AD* can be that of the resultant.



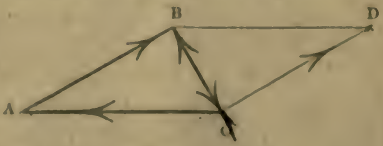
Secondly. To prove that the diagonal of the parallelogram represents the magnitude of the resultant also, when the sides respectively represent the component forces. Let *P*, *Q*, *R*, acting in the directions of the arrows, as in the figure, keep the point *A* at rest. Let the lines *AD*, *AB*, *AF*, represent respectively the forces. Complete the parallelograms *AC*, *AE*, and draw the diagonals. The resultant of any two of the forces must be equal in magnitude, and opposite in direction, to the third force, since there is equilibrium: therefore, *CAF*, *BAE*, are straight lines. *AE* is parallel to *CD*, and *AC* is parallel to *DE;* and *ACDE* is a parallelogram, of which the side *AC* is equal to the side *DE*. But *DE* by construction is equal to *AF;* therefore *AD* and *AB* representing the forces *P* and *Q*, *AC* represents a force equal in magnitude to the third force *R*, and opposite in direction; and thus represents the resultant of *P* and *Q* in magnitude as well as direction.

5. Prop. *If three forces acting at a point keep it in equilibrium, and a triangle be formed by three lines drawn in their directions, the sides of the triangle, taken in order, will represent the forces. Conversely, if three forces which act at a point be represented by the sides of a triangle, taken in order, they will be in equilibrium.* This proposition is called the *triangle of forces.*

If the three forces, *P*, *Q*, *R*, in equilibrium, act at the point *A*, as in the figure, the triangle *ABC*, which is half the parallelogram formed on the lines representing *P* and *Q*, will have its side *BA* representing the force *R*, by taking the sides in order, as shewn by the direction of the arrows: for *AB* represents the resultant of *P* and *Q*, when taken in the opposite direction.



Conversely, If the three forces were represented by the sides of the triangle *ABC* taken in order, we might form a parallelogram with any one of the sides, as *BC*, for its diagonal, and the resultant of the other two forces, represented by *CA* and *CD* (or *AB*), would be represented by this diagonal taken in the opposite direction, which would make equilibrium with the third force.



By means of this proposition we can resolve a given force into two others which are equivalent to it, in any given directions.

6. Prop. *If three forces acting at a point are in equilibrium, they are proportional each to the sine of the angle contained between the other two.*

In figure art. 5, the sides of the triangle ABC represent the forces P, Q, R, respectively; or,

$$
\begin{aligned}
P : Q : R &:: AC : CB : BA \\
&:: \sin. ABC : \sin. BAC : \sin. ACB \\
&:: \sin. BAQ : \sin. BAC : \sin. PAQ \quad \checkmark \\
&:: \sin. QAR : \sin. PAR : \sin. PAQ
\end{aligned}
$$

Since the sines of the angles equal the sines of their supplements.

7. Prop. *If P and Q be two forces which act at a point, the angle between their directions being θ, then if R equals their resultant, we have* $R^2 = P^2 + Q^2 + 2 \,.\, P \,.\, Q$ cosine θ.

By trigonometry we have in triangle ABC, art. 5.

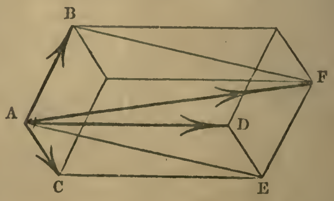
$$AB^2 = AC^2 + BC^2 - 2\, AC \,.\, BC \,.\, \cos. ACB$$

$$\text{and } \cos. ACB = -\cos. BCP = -\cos. \theta.$$

$$\therefore R^2 = P^2 + Q^2 + 2\, P . Q \cos. \theta.$$

8. Prop. *If three forces acting at a point in different planes be represented in direction and magnitude by the three edges of a parallelopiped, then the diagonal will represent their resultant in direction and magnitude; and reciprocally, if the diagonal represents a force, it is equivalent to three forces represented by the edges of the parallelopiped.*

Let the three edges AB, AC, AD of the parallelopiped in the figure represent the three forces. Then AE, the diagonal of the face $ACED$, represents the resultant of the forces AC and AD. Compounding this with the third force represented by AB, we have AF, the diagonal of the

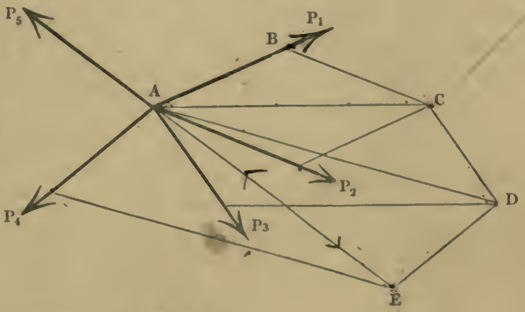


parallelogram *AEFB*, representing the resultant of *AE* and *AB*, or of the forces *AC*, *AD*, *AB*.

Reciprocally, the force *AF* is equivalent to the components *AB*, *AE*, or to the component forces, *AB*, *AC*, and *AD*.

9. Prop. *If any number of forces are represented in direction and magnitude by the sides of a polygon taken in order, they will, when applied at one point, produce equilibrium.*

Let P_1, P_2, P_3, P_4, P_5, be forces acting at the point *A*, as in the figure.



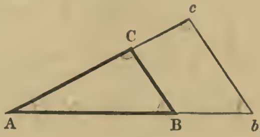
Let P_1 be represented by *AB*.
,, P_2 ,, the parallel line *BC*.
,, P_3 ,, ,, *CD*.
,, P_4 ,, ,, *DE*.
,, P_5 ,, ,, *EA*.

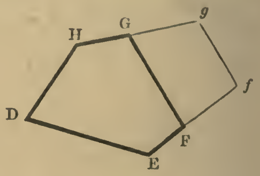
Or, let the forces be represented in magnitude and direction by the sides of the polygon *ABCDE*, taken in order. If we complete the parallelograms *AC*, *AD*, *AE*, and draw the diagonals *AC*, *AD*, we see that *AC* represents the resultant of forces P_1 and P_2; compounding this resultant with the force P_3, we see that *AD* represents their resultant, or the resultant of P_1, P_2, and P_3; compounding this last resultant with the force P_4, we see that their resultant is represented by the line *AE*, acting in the direction from *A* to *E*, which would consequently balance the last force P_5, represented by the last side *EA* of the polygon, and acting in the direction from *E* to *A*. It will be seen from the proof that it is not necessary the forces should lie all in one plane.

This proposition is called the *polygon of forces*.

When three forces act in equilibrium at a point, any three lines taken parallel to their directions will form a triangle, the sides of which respectively will represent the forces; but when there are four or more forces, this will not hold, because the relation which subsists between the sides and angles of triangles does not hold in four-sided figures or polygons.

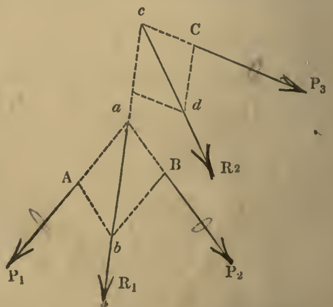
For instance, in the figures, if *CB* and *cb* be parallel, and also *FG* and *fg*, the triangles *ABC*, *Abc*, being similar and similarly situated, the sides being respectively proportional, would represent the same three forces; but, although the sides of the polygon *DEFGH* might represent a system of five forces in direction and magnitude, yet they could not be represented by the sides of the polygon *DEfgH* in magnitude also.





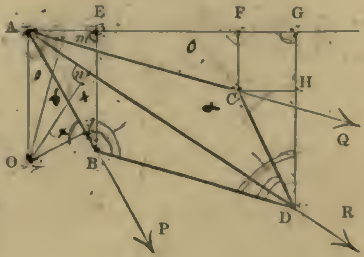
10. When any number of given forces act at given points in a plane, we may find, *graphically*, the magnitude and direction of the resultant of the system.

Let P_1, P_2, P_3, in the figure, have their points of application A, B, C. Producing the directions of the forces P_1 and P_2, until they meet at the point a, we form the parallelogram ab upon the lines representing them, and the diagonal is their resultant R_1 in the figure. Producing R_1, until it meets at c the direction of the force P_3, and drawing the parallelogram cd on the lines representing R_1 and P_3, we have the diagonal representing R_2 their resultant, or the resultant of P_1, P_2, and P_3. By pursuing the same method we may find the resultant for any number of forces.



Definition. The *moment* of a force about any point is the product of the force into the perpendicular let fall from the point upon the direction of the force. The moment, as we shall see in the next chapter, measures the tendency of the force to produce rotatory motion about the fixed point.

11. Prop. *The moment of the resultant about any point in the plane of the forces equals the sum of the moments of the forces.*



Let the forces P and Q, acting at A, be represented by the lines AB, AC, and their resultant R by AD, the diagonal of the parallelogram drawn upon AB, AC. Let O be the point about which the moments are taken; join OA, and draw $AEFG$ perpendicular to OA; draw Ol, Om, On, respectively perpendicular to AB, AC, AD; and BE, CF, DG, perpendiculars to $AEFG$; and CH parallel to that line.

Now the triangles OlA, OmA, OnA, are respectively similar to the triangles AEB, AFC, AGD.

$$\text{Whence } \frac{AE}{AB} = \frac{Ol}{OA} \text{ or } AE = \frac{AB \,.\, Ol}{OA}$$

$$\frac{AF}{AC} = \frac{Om}{OA} \text{ or } AF = \frac{AC \,.\, Om}{OA}$$

$$\frac{AG}{AD} = \frac{On}{OA} \text{ or } AG = \frac{AD \,.\, On}{OA}$$

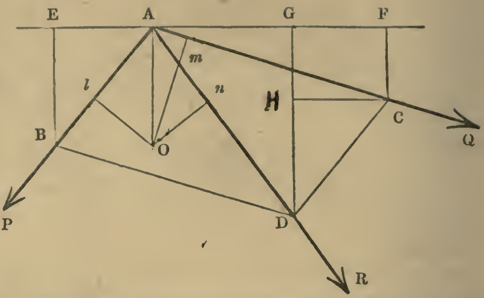
$$\text{but } AE = CH = FG \;\therefore\; AF + AE = AG$$

$$\text{or, } AB \,.\, Ol + AC \,.\, Om = AD \,.\, On$$

$$\text{or, } P \,.\, Ol + Q \,.\, Om = R \,.\, On$$

Or the sum of the moments of the components equals the moment of the resultant about any point in their plane, or about an axis perpendicular to their plane.

If the point O fell within the angle formed by the forces, we should have the moment of one of the forces tending to cause ro-



tation in the opposite direction to the other, and it must then be considered as negative if the other were positive, and the algebraic sum of the moments of the components still equals that of the resultant. In the annexed figure we have the proof the same as above, except that $AG = AF - AE$,

$$\text{and } AC \,.\, Om - AB \,.\, Ol = AD \,.\, On$$
$$\text{or } Q \,.\, Om - P \,.\, Ol = R \,.\, On.$$

By compounding R with another force acting at A, we should obtain a like result: or the moment of the resultant of these forces acting at A equals the algebraic sum of the moments of the forces. The proposition may, in the same manner, be extended to any number of forces acting at a point.

EXAMPLES.

1. Shew that if θ be the angle between two forces of given magnitudes, their resultant is the greatest when $\theta = 0$, least when $\theta = \pi$, and intermediate for intermediate values of θ. If the component forces be P and Q, what is the magnitude of the resultant when $\theta = 0$, and also when $\theta = \pi$? *Ans.* $(P + Q)$ *and* $(P - Q)$.

2. If two equal forces (P) meet at an angle of 60°, shew that their resultant $= P\sqrt{3}$.

3. If two equal forces meet at an angle of 135°, shew that their resultant $= P(2 - \sqrt{2})^{\frac{1}{2}}$.

4. If three forces, whose magnitudes are $3m$, $4m$, and $5m$, act at one point and are in equilibrium, shew that the forces $3m$ and $4m$ are at right angles to each other.

5. If two equal forces are inclined to each other at an angle of 120°, shew that their resultant is equal to either of them.

6. If the magnitudes of two forces are 6 and 11, and the angle between their directions 30°, shew that the magnitude of their resultant is 16·47 nearly.

7. Shew, that in the last question the resultant makes with the force 6 the angle whose sine is ·3339, and with the force 11 the angle whose sine is ·1821, which are the sines of 19° 30′ and 10° 30′ nearly.

8. Apply the proof of the *polygon of forces* to the case of five equal forces represented by the sides of a regular pentagon taken in order.

9. Enunciate all the propositions requisite to prove that the resultant is in every respect mechanically equivalent to the component forces.

10. A cord PAQ is tied round a pin at the fixed point A, and its two ends are drawn in different directions by the forces P and Q: shew that the angle between these directions is found from the expression $\cos. \theta = -\frac{3(P^2+Q^2)-2PQ}{8PQ}$ when the pressure on the pin is equal to $\frac{P+Q}{2}$.

11. A cord whose length is $2l$ is tied at the points A and B in the same horizontal line, whose distance is $2a$: a smooth ring upon the cord sustains a weight w: shew that the force of tension in the cord $= \dfrac{w}{2\sqrt{1-\frac{a^2}{l^2}}}$

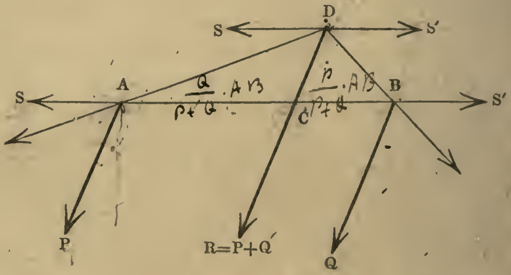
CHAPTER II.

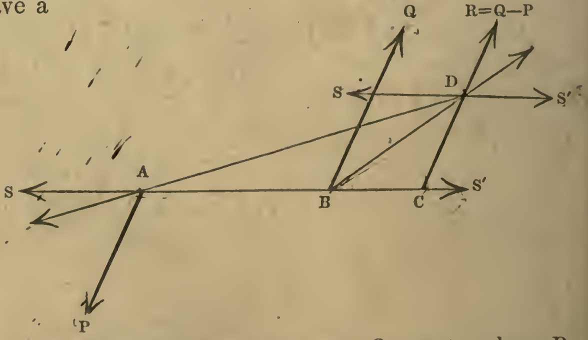
ON FORCES WHOSE DIRECTIONS ARE PARALLEL.

Though the propositions in the last chapter will not apply at once to forces acting at different points, of which the directions are parallel, yet we can reduce the proof of the method of finding their resultant to that of two forces acting at one point in different directions.

12. Prop. *If two parallel forces P and Q act at points A and B respectively, then their resultant equals their algebraic sum in magnitude, and acts at a point C in the same straight line with A and B, such that* $P \times AC = Q \times BC$.

Let the forces act as in the figures at A and B; at these points apply equal and opposite forces, S and S': they will not affect the system. P and S at A will have a resultant in the direction AD; Q and S' at B will have a resultant in the direction BD; and in the lower figure, where the forces act in opposite directions, we suppose Q greater than P, so that the resultant of Q and S' will lie nearer Q than the re-





sultant of P and S does to P; and therefore the directions of the resultants will meet at some point, as D, in both figures. We may suppose the whole of the forces to act at the point D. Resolving the forces parallel to the original directions, we shall have forces S and S' parallel to AB, which will destroy each other; and forces P and Q acting parallel to their original directions, giving a resultant $R = P + Q$ in the upper figure, and $R = Q - P$ in the lower figure. To find the point C in the line AB, or AB produced, where R acts, we have,

$$\text{from triangle } ACD, \quad P : S :: CD : AC$$
$$\text{from triangle } BCD, \quad S' : Q :: BC : CD$$

Compounding these ratios, we have

$$P : Q :: BC : AC$$
$$\text{or, } P \times AC = Q \times BC$$

13. If AB be perpendicular to the direction of the forces, AC and BC are called the *arms* of the forces, and the products $P.AC$, $Q.BC$, are the *moments* of the forces, about the point C.) If the forces are inclined at an angle α to AB, then the perpendiculars from C on the lines of action of the forces are $AC.\sin.\alpha$ and $BC.\sin.\alpha$, and the moments are $P.AC.\sin.\alpha$ and $Q.BC.\sin.\alpha$.

14. If C be a fixed point, its resistance will destroy the effect of the resultant force; so that P and Q will be in equilibrium about such a point, when their moments, tending to cause rotation opposite ways about it, are equal to each other.

15. To find AC in terms of AB, we have

$$P \times AC = Q(AB - AC), \text{ in the upper figure;}$$
$$\text{or, } AC = \frac{Q}{P+Q} AB \quad \text{and similarly, } BC = \frac{P}{P+Q} AB$$
$$\text{and } P \times AC = Q(AC - AB), \text{ in the lower figure;}$$
$$\text{or, } AC = \frac{Q}{Q-P} AB \quad \text{and similarly, } BC = \frac{P}{Q-P} AB$$

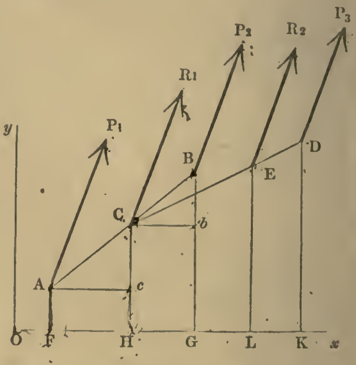
The point C is determined in both cases; unless in the latter $P = Q$, when $AC = $ infinity; but then the resultant $R = Q - P = 0$. This is a peculiar case; the effect of two equal and parallel forces which do not act at the same point being to produce rotatory motion only. Such forces constitute what is called a *statical couple*, and all tendency to rotatory motion can be referred to forces forming such *couples*. If the forces are inclined at an angle α to AB, then $P.AB.\sin.\alpha$ is the moment of the couple.

16. PROP. *To find the resultant of any number of parallel forces which act at any points in one plane.*

First, let the parallel forces P_1, P_2, P_3, &c. . . . P_n, have their points of application A, B, D, &c. and act towards the same part. Take any two lines, Ox, Oy, at right angles to each other; join AB, and let C be the point of application of the resultant, R_1, of P_1 and P_2.

Then $R_1 = P_1 + P_2$

and $P_1 \times AC = P_2 \times BC$



Draw the lines AF, BG, CH, DK, parallel to Oy; and Ac, Cb, parallel to Ox. The triangles ACc, CBb, are similar, and

$$\frac{AC}{Cc} = \frac{BC}{Bb}$$ AC : Cc :: BC : Bb

$$\therefore P_1 \times Cc = P_2 \times Bb$$

$$\text{or, } P_1 \times (CH - AF) = P_2 \times (BG - CH)$$

$$(P_1 + P_2) \times CH = P_1 \times AF + P_2 \times BG$$

$$\text{or, } R_1 \times CH = P_1 \times AF + P_2 \times BG$$

Taking another force, P_3, and compounding with R_1 acting at C, we find the second resultant, $R_2 = R_1 + P_3 = P_1 + P_2 + P_3$,

$$\text{and } R_2 \times EL = R_1 \times CH + P_3 \times DK$$
$$= P_1 \times AF + P_2 \times BG + P_3 \times DK$$

and so onwards for any number of forces.

If we put $AF = y_1$, $BG = y_2$, $DK = y_3$, &c.

$OF = x_1$, $OG = x_2$, $OK = x_3$, &c.

or, if $x_1 y_1$, $x_2 y_2$, $x_3 y_3$, &c. . . . $x_n y_n$ are the co-ordinates of the points A, B, D, &c., the points of application of the n forces, the above formula becomes

$$R_2 \times EL = P_1 . y_1 + P_2 . y_2 + P_3 . y_3$$

and if $\bar{x}$, $\bar{y}$, were the co-ordinates of the point of application of R, the resultant of the n forces, we should have

$$R = P_1 + P_2 + P_3 + \&c. \; . \; . \; . \; + P_n$$

$$\text{and } R . \bar{y} = P_1 . y_1 + P_2 . y_2 + P_3 . y_3 + \&c. \; . \; . \; . \; + P_n . y_n$$

If we had drawn lines parallel to Ox, we should have found, in the same way,

$$R . \bar{x} = P_1 . x_1 + P_2 . x_2 + P_3 . x_3 + \&c. \; . \; . \; . \; + P_n . x_n$$

These formulæ are often written more concisely by using the Greek letter Σ as the sign of summation; and P being any one of the forces, x, y, being the co-ordinates of its point of application, then

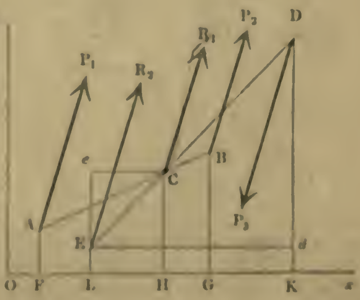
$$R . \bar{x} = \Sigma (P . x)$$
$$R . \bar{y} = \Sigma (P . y)$$
$$R = \Sigma (P)$$

The point whose co-ordinates are $\bar{x}$, $\bar{y}$, is called *the center of parallel forces.* It depends on the magnitude of the forces and their points of application; but is independent of the angle which their direction makes with any given line.

Secondly. When some of the forces act in opposite directions, they must be taken negative; and so also when the co-ordinates of the points of application are negative, they must be applied with their proper signs; and then the above formulæ will apply to all cases.

If R_1, the resultant of P_1 and P_2, as found in the previous case, be compounded with a force P_3 acting as in the figure at D, by joining $C\,D$, and producing it in the direction of the greater force, say R_1, we have R_2 the second resultant $= R_1 - P_3$; and E being its point of application,

$$R_1 \times EC = P_3 \times ED$$



Drawing Ce, Ed, parallel to Ox and meeting DK in d, EL produced in e, the triangles ECe, EDd, are similar, and

$$\frac{EC}{Ee} = \frac{DE}{Dd}$$

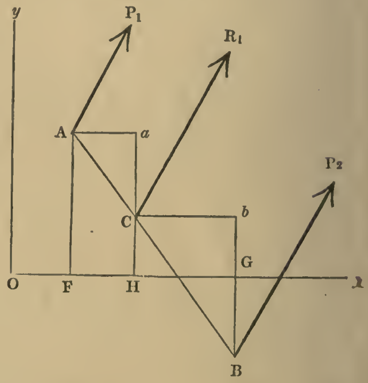
$$\therefore\ R_1 \times Ee = P_3 \times Dd$$

$$\text{or, } R_1(CH - EL) = P_3(DK - EL)$$

$$\text{or, } (R_1 - P_3) \,.\, EL = R_1 \times CH - P_3 \times DK$$

$$\text{or, } R_2 \times EL = P_1 \times AF + P_2 \times BG - P_3 \times DK$$

and so for any other forces acting in the opposite direction to P_1 and P_2, &c.



Again. Let y_2, the ordinate of the point of application of P_2, be negative $= BG$ in figure; draw Aa, Cb, parallel to Ox.

$$P_1 \times AC = P_2 \times CB$$

or, $P_1 \times Ca = P_2 \times Bb$ by similar triangles.

$$P_1(AF - CH) = P_2(CH + BG)$$

$$(P_1 + P_2)\,CH = P_1 \times AF - P_2 \times BG$$

or, $R_1 . \bar{y} = P_1 . y_1 - P_2 . y_2$, as we should have found by putting y_2 with its proper sign in the general formula, which thus applies generally to all cases of parallel forces.

When some of the forces are negative and others positive, we may have the sum $\Sigma(P) = 0$;

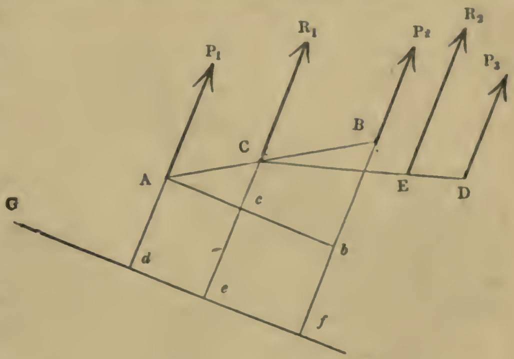
$$\text{or, } R = 0$$

and the system of forces may be equivalent to a *couple*.

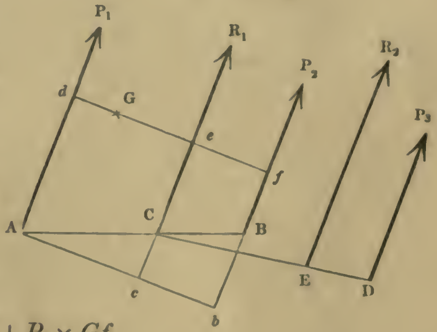
17. Prop. *The algebraic sum of the moments of any number of parallel forces which act in one plane about any point in the plane, equals the moment of their resultant about that point.*

Let P_1, P_2, P_3, be parallel forces acting at the points A, B,

D, respectively; R_1 and R_2 the resultants, as before; G the point about which the moments are taken, and called *the center of moments.*



Draw $Gdef$ and Acb perpendiculars to the direction of the forces in both figures. In the first figure both P_1 and P_2 will tend to cause rotation the same way round G; but in the other figure they tend opposite ways, and must therefore be taken with different signs.



In the first figure, the sum of the moments of P_1 and P_2 about $G = P_1 \times Gd + P_2 \times Gf$

$$= P_1 (Ge - de) + P_2 (Ge + ef)$$

$$= (P_1 + P_2) Ge + P_2 \times cb - P_1 Ac \quad . \; . \; . \text{ since } cb = ef,\ de = Ac$$

$$= R_1 \times Ge + (P_2 \times CB - P_1 \times AC) \cos. BAb$$

$$= R_1 \times Ge = \text{moment of the resultant about } G$$

since $P_1 \times AC = P_2 \times CB$ when C is the point of application of the resultant.

In the second figure, the algebraic sum of the moments about G

$$= P_2 \times Gf - P_1 \times Gd$$

$$= P_2 (Ge + ef) - P_1 (de - Ge)$$

$$= (P_1 + P_2) Ge + P_2 \times ef - P_1 \times de$$

$$= R_1 \times Ge \quad . \; . \; . \text{ as before, since } P_2 \times ef = P_1 \times de$$

If we take another force, P_3, we find, in the same way, the moment of the second resultant, R_2, equals the algebraic sum of the moments of R_1 and P_3, or of P_1, P_2, and P_3, and so for any number of forces.

EXAMPLES.

1. Two parallel forces acting in the same direction have their magnitudes 5 and 13, and their points of application 6 feet distant; shew that their resultant acts at a point $4\frac{1}{3}$ feet from the point of application of the force 5, and $1\frac{2}{3}$ feet from that of the force 13. What is the magnitude of the resultant?

2. If the forces in the last question act in opposite directions, shew that the point of application of their resultant is distant $3\frac{3}{4}$ feet from the point of application of force 13, and $9\frac{3}{4}$ feet from that of force 5. What is the magnitude of the resultant?

3. If two parallel forces, P and Q, act in the same direction at the points A and B, and make an angle θ with the line AB, shew that the moment of each of them about the point of application of their resultant $= \dfrac{P.Q}{P+Q} AB \sin. \theta$.

4. If three forces which act at *a point* be represented in direction and magnitude by the sides of a triangle taken in order, they will make equilibrium; shew that if, instead of acting at one point, they each act in the line which is the side of the triangle representing it, they are equivalent to a *statical couple*.

5. If three equal parallel forces act at the three angles of an equilateral triangle, shew that *their center*, or point of application of their resultant, is in the line drawn from any angle to the middle of the opposite side, and at $\frac{2}{3}$ the length of the line measured from the angle: being independent of the angle which the forces make with the plane of the triangle.

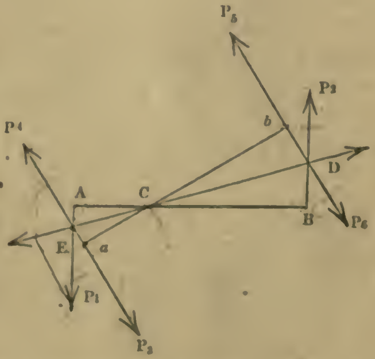
6. In question 4, if a, b, c, be the sides of the triangle opposite to the angles A, B, C, respectively, then the moment of the couple equals $ab \sin. C = ac \sin. B = bc \sin. A$.

CHAPTER III.

ON THE THEORY OF COUPLES.

We saw in the last chapter that two equal and parallel forces acting in opposite directions and at different points of a body had no single resultant, but constituted a *statical couple*, tending to produce rotatory motion. This tendency can be balanced only by an equal and opposite tendency produced by an opposite couple. Statical couples have peculiar properties, which we will now discuss, and chiefly by employing the *super-position of equilibrium*. See page 6.

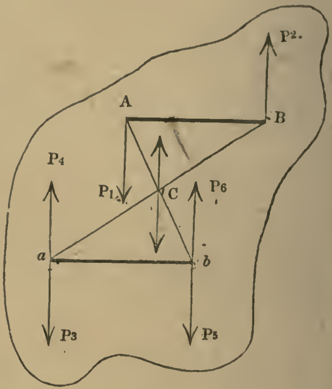
18. Prop. *A couple may be turned round in any manner in its own plane without altering its statical effect.*



Let P_1ABP_2 be the original couple; take $ab=AB$, and turned round any point C; apply equal and opposite forces, P_3, P_4, perpendicular to ab at a; and similarly, P_5 and P_6 at b; these will not affect the system, being in equilibrium amongst themselves: let each of them equal P_1 or P_2. Then P_1 at A, and P_4 at a, are equivalent to a force bisecting the angle P_1EP_4 between them, or a force in CE; similarly, P_2 and P_6 are equivalent to an equal force in CD. These forces being equal and opposite may be removed; that is, we may remove from the system the forces P_1, P_2, P_4, P_6, and we have remaining the forces P_3 and P_5 at a and b, forming the couple P_3abP_5, which is the same as if we had turned the original couple round the point C until its arm came to the position ab.

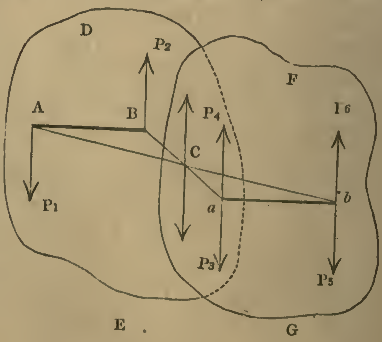
19. Prop. *A couple may be removed to any other part of its own plane, its arm remaining parallel to the original direction; and it may be removed to any other plane, in the body on which it acts, parallel to its own plane, its arm being parallel to the original arm.*

First. Let the arm AB of the original couple P_1ABP_2 be removed in its own plane to the parallel position ab; let forces P_3, P_4, P_5, P_6, each equal to the original forces, be applied perpendicularly to ab at the extremities a and b in opposite pairs, as in the figure.



Join Ab and aB, these lines will bisect each other in C; and P_1 at A, and P_5 at b, are equivalent to a force $=2P_1$ at C, parallel to the original direction; similarly, P_2 at B, and P_4 at a, are equivalent to a force $=2P_2$ at C, opposite to the former; these will consequently balance each other, and may be removed, or the forces P_1, P_2, P_4, P_5, may be removed, and we have remaining the couple P_3abP_6, equivalent to the original couple removed parallel to itself in its own plane.

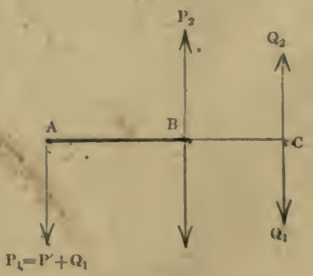
Secondly. Let the arm AB of the original couple be removed from its own plane DE, parallel to itself, to ab in the parallel plane FG; let equal forces, each equal to P_1 or P_2, be applied at a and b, as in the former case. Join Ab and Ba; these lines will bisect each other in C. The forces P_1 and P_5 will be equivalent to a force $=2P_1$ at C, parallel to the original di-



rection, and P_2 and P_4 will be equivalent to an equal and opposite force at the same point; these equal and opposite forces at C may be removed, and there remains the couple $P_3 a b P_6$, equivalent to the original couple removed to the plane FG.

20. Prop. *All statical couples are equivalent to each other whose planes are parallel and moments equal.*

Definition. The moment of a couple is the product of one of the forces into the arm, or $P.AB$ in the foregoing propositions.



Let $P_1 A B P_2$ be the original couple, whose moment is $P.AB$. Produce AB to C, and apply there equal and opposite forces Q_1 and Q_2, such that $Q.AC=P.AB$. We may suppose P_1 at A to be the sum of two forces, P' and Q_1; now

$$P_1.AB=Q_1.AC=Q_1(AB+BC).$$
$$\therefore (P_1-Q_1)AB=Q_1.BC$$
$$=P'.AB$$

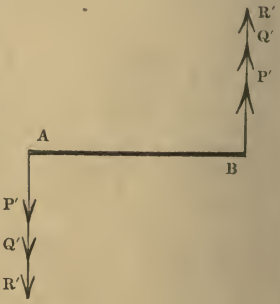
and forces P' at A, and Q_1 at C, have a resultant parallel to their directions and equal to their sum at B. This force, $P'+Q=P_1$, will balance P_2, and therefore P' at A, Q_1 at C, and P_2 at B, may be removed; and there remains the couple $Q.AC$, which is therefore equivalent to the original couple.

By the previous propositions this couple, $Q.AC$, may be removed into any plane parallel to its own, and turned round in any manner in that plane.

Definition. The *axis of a couple* is a line perpendicular to the plane of the couple; and its length being taken proportional to the moment of the couple, represents it in magnitude. The tendency to rotatory motion being round the axis, and the length of the axis representing this tendency as measured by the moment, the axis represents completely the couple. The effect of the previous propositions is consequently this: that the axis, *of fixed length*, may be removed any where within the body acted on, parallel to itself.

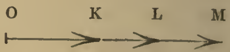
When any number of couples act upon the body, they can be compounded into one resultant couple.

21. Prop. *When any number of couples act upon a body in parallel planes, the moment of the resultant couple equals the sum of the moments of the component couples.*



Let P, Q, R, &c. be the forces; a, b, c, &c. their arms respectively; the couples can be removed all into one plane, turned round, and moved in that plane, and their arms changed to a common arm, whilst their moments remain unchanged. Let m be the common arm AB; P', Q', R', &c., the forces; so that $P'.m=P.a$, $Q'.m=Q.b$, $R'.m=R.c$, &c.; but the forces P', Q', R', &c. at A are equivalent to a force

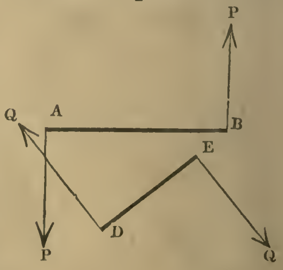
$$P'+Q'+R'+\&c.=\frac{P.a}{m}+\frac{Q.b}{m}+\frac{R.c}{m}+\&c.$$



And similarly, the forces at B are equal to the same sum; and the moment of the resultant couple $=(P'+Q'+R'+\&c.)AB$

$$=(P'+Q'+R'+\&c.)\,m$$
$$=Pa+Qb+Rc+\&c.$$

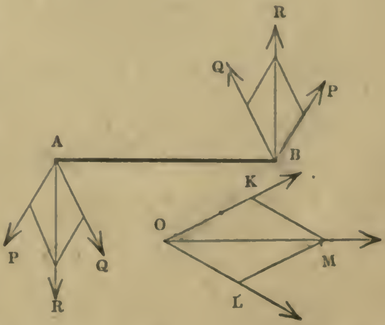
This is the same thing as taking the algebraic sum of the axes, as OK, KL, LM, &c. for the resultant axis, when the component axes are parallel. If any of the couples tend to cause rotation the contrary way round, we must take them with contrary signs, or their axes must have been measured in the opposite direction from O. An axis is therefore balanced by an equal and opposite axis, or a couple by an equal and opposite couple. If $PABP$, $QDEQ$, were couples whose moments were equal and



opposite, or $P.AB = -Q.DE$, they would evidently make equilibrium with each other.

22. Prop. *If two sides of a parallelogram represent the axes of two component couples, the diagonal represents the axis of the resultant couple.*

Let OK, OL, be the axes of the component couples, then OM, the diagonal of the parallelogram formed on them, represents the axis of a couple equivalent to them.



The planes of the couples being perpendicular to their axes respectively, will be inclined at the same angle to each other as the axes themselves are. The couples can be moved and turned round, each in their own plane, until their arms are in the intersection of their planes; and their moments being kept the same, they can be brought to have the same arm. Let AB be this common arm in the intersection of the planes; $PABP$, $QABQ$, the couples. Completing the parallelograms on the lines representing P and Q respectively, the diagonal represents their resultants R, at A and B, and the two couples are equivalent to a couple $RABR$.

If θ be the angle between P and Q, or between OK and OL, $R^2 = P^2 + Q^2 + 2PQ \cos.\theta$ . . . from the triangle of forces.

$$\begin{aligned} \therefore R.AB &= AB.\sqrt{P^2 + Q^2 + 2PQ\cos.\theta} \\ &= \sqrt{P^2.AB^2 + Q^2.AB^2 + 2P.AB \times Q.AB\cos.\theta} \\ &= \sqrt{OK^2 + OL^2 + 2OK.OL\cos.\theta} \\ &= OM \end{aligned}$$

And OK, OL being respectively perpendicular to the planes of the couples $PABP$, $QABQ$, we have OM perpendicular to the plane of the resultant couple $RABR$; therefore OM represents the axis of the resultant couple. Let L and M be the moments of the component couples, G the moment of their resultant; then

$$G^2=L^2+M^2+2L.M.\cos.\theta.$$

Cor. If L, M, N, were the axes of three component couples, we could shew, as in article 8, that G, the axis of the resultant couple, would be represented in direction and magnitude by the diagonal of the parallelopiped formed upon them.

If L, M, N, were at right angles each to the plane passing through the other two, the parallelopiped would be rectangular, and we should have

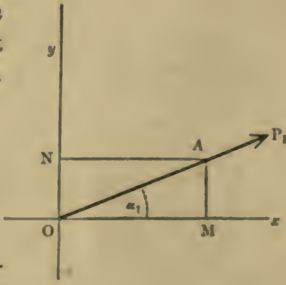
$$G^2=L^2+M^2+N^2.$$

CHAPTER IV.

ON ANALYTICAL STATICS IN TWO DIMENSIONS.

In this chapter we refer the points of application and the directions of the forces to co-ordinate axes Ox, Oy, at right angles to each other.

23. Prop. *Required the magnitude and direction of the resultant of any number of forces in one plane acting at one point, and the conditions in order that there may be equilibrium.*



Let P_1, P_2, P_3, &c. . . . P_n, be the n forces, and let the point at which they act be taken for the origin of co-ordinates.
Let P_1 make the angle α_1 with Ox
P_2 - - - α_2 - -
P_3 - - - α_3 - -
&c. - - - &c.
P_n - - - α_n - -

Let OA represent the force P_1; complete the right-angled parallelogram $OMAN$, then OM represents the resolved part of P_1 in Ox, ON that in Oy.

Let X_1, X_2, X_3, &c. . . . X_n, be the resolved parts of the forces respectively in Ox,

Y_1, Y_2, Y_3, &c. . . . Y_n . . . in Oy,

we have $OM = X_1 = P_1 \cos. \alpha_1$ $\qquad ON = Y_1 = P_1 \sin. \alpha_1$

and similarly, $X_2 = P_2 \cos. \alpha_2$ $\qquad Y_2 = P_2 \sin. \alpha_2$

$X_3 = P_3 \cos. \alpha_3$ $\qquad Y_3 = P_3 \sin. \alpha_3$

&c. &c. $\qquad$ &c. &c.

$X_n = P_n \cos. \alpha_n$ $\qquad Y_n = P_n \sin \alpha_n$.

But the components in Ox are equivalent to a single force

$$= X_1 + X_2 + X_3 + \&c. \ldots + X_n.$$

This sum we often write more concisely by employing the Greek letter Σ as the sign of summation;

$$\text{or, } X_1+X_2+X_3+\&c. \ldots +X_n=\Sigma(X)$$

In the same way, in the axis of y we have

$$Y_1+Y_2+Y_3+\&c. \ldots +Y_n=\Sigma(Y)$$

$$\text{or we have} \qquad \Sigma(X)=\Sigma(P \cos. \alpha)$$

$$\Sigma(Y)=\Sigma(P \sin. \alpha)$$

Let R be the resultant required, θ the angle it makes with Ox. The resolved parts of R in the axes must equal the resolved parts of the forces in the same directions; or,

$$R \cos. \theta=\Sigma(X)$$

$$R \sin. \theta=\Sigma(Y)$$

$$\therefore \tan. \theta=\frac{\Sigma(Y)}{\Sigma(X)} \qquad (1)$$

$$R^2=R^2(\sin.^2\theta+\cos.^2\theta)$$

$$=(\Sigma(X))^2+(\Sigma(Y))^2 \qquad (2)$$

These equations (1) and (2) give the magnitude and direction of the resultant.

If the forces are in equilibrium, their resultant $=0$, and $R=0$ gives

$$0=(\Sigma(X))^2+(\Sigma(Y))^2$$

But square quantities being essentially positive, this equation cannot be true unless we have

$$\Sigma(X)=0 \qquad \Sigma(Y)=0$$

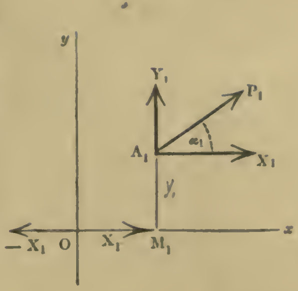
These are the two necessary and sufficient equations for equilibrium, when any number of forces in one plane act at one point.

24. Prop. *To investigate the expressions for the resultant force and resultant couple, and to find the conditions of equilibrium, when any number of forces act at various points of a rigid body in one plane.*

Let Ox, Oy, be the co-ordinate axes, and the plane passing through them be that in which the forces P_1, P_2, &c. . . . P_n, act.

Let α_1, α_2, &c. . . . α_n, be the angles they make with Ox respectively.

Let x_1, y_1, be the co-ordinates OM_1, A_1M_1 of A_1 the point of application of the force P_1, and let x_2, y_2, &c. . . . x_n, y_n, be the co-ordinates of the points of application of the others respectively.



Resolving P_1 in the directions of Ox and Oy, we shall have components

$$X_1 = P_1 \cos. \alpha_1 \qquad\qquad Y_1 = P_1 \sin. \alpha_1$$

at the point A_1.

Apply at O, in Ox, equal and opposite forces, each equal to X_1, which will not affect the system; and X_1 at A_1, with $-X_1$ at O, form a couple, with its arm A_1M_1.

Then X_1 at A_1 is equivalent to a force X_1 at O, and a couple $X_1 \times A_1M_1 = X_1y_1$.

Similarly, applying at O in Oy forces equal and opposite, and each equal to Y_1, we have Y_1 at A_1 equivalent to Y_1 at O, and a couple

$$-Y_1 \times OM_1 = -Y_1x_1$$

This last couple will be negative if the former be taken positive, as tending to cause rotation the contrary way. We consider those couples *positive* which tend to cause rotation in the *direction of the hands of a watch.*

Proceeding in the same way with all the other forces, we shall have a sum of forces

$$X_1 + X_2 + X_3 + \&c. \ldots + X_n = \Sigma(X)$$

at O in Ox;

$$Y_1 + Y_2 + Y_3 + \&c. \ldots + Y_n = \Sigma(Y)$$

at O in Oy;

and couples $X_1y_1 + X_2y_2 + X_3y_3 + \&c. \ldots + X_ny_n = \Sigma(Xy)$

$$-Y_1x_1 - Y_2x_2 - Y_3x_3 - \&c. \ldots - Y_nx_n = \Sigma(-Yx)$$

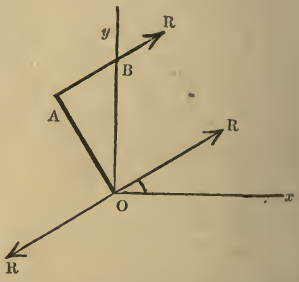
The couples being in the same plane, we have their resultant axis G equal to the algebraic sum of the component axes; or,

$$G = \Sigma(Xy) - \Sigma(Yx)$$
$$= \Sigma(Xy - Yx) \qquad (1)$$

If R be the resultant force acting at O, θ the angle it makes with Ox,

$$R \text{ cos. } \theta = \Sigma(X)$$
$$R \text{ sin. } \theta = \Sigma(Y)$$
$$\text{tan. } \theta = \frac{\Sigma(Y)}{\Sigma(X)} \qquad (2)$$
$$\text{and } R^2 = (\Sigma(X))^2 + (\Sigma(Y))^2 \qquad (3)$$

The equations (1), (2), and (3), determine G and R; they can, however, be simplified when neither G nor $R=0$. The moment of the couple remaining the same, let its forces be each made equal to R; then let it be moved and turned round until one of its forces acts at O in an opposite direction to the resultant force; AO being the arm. These two forces balancing, may be removed, and there remains the other force of the couple acting in ABR in the figure. This final resultant being parallel to R, makes the angle θ with the axis of x.



To find the equation of the line ABR, we have OA the arm of the couple $= \frac{G}{R}$; and $OB = \frac{OA}{\text{cos. } \theta} = \frac{G}{R \text{ cos. } \theta}$

$$\text{or, } OB = \frac{G}{\Sigma(X)}$$

and equation of the line ABR is

$$y = \text{tan. } \theta . x + OB$$
$$\text{or, } y = \frac{\Sigma(Y)}{\Sigma(X)} . x + \frac{G}{\Sigma(X)}$$

When there is equilibrium, we must have both the resultant force $=0$ and the resultant couple $=0$. These conditions give us

$$\Sigma(X) = 0$$
$$\Sigma(Y) = 0$$
$$\Sigma(Xy - Yx) = 0$$

These are *the three necessary and sufficient equations of equilibrium,* when any forces act on a free body in one plane.

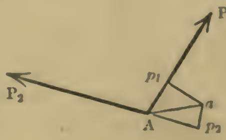
25. If there were a fixed point in the plane of the forces, we might take it for the origin of co-ordinates O, and its resistance would destroy the effect of the resultant force R, and we should have the condition of equilibrium only $G=0$;

$$\text{or, } \Sigma\,(Xy-Yx)=0$$

or there must be no tendency to rotation around the fixed point.

26. PROP. *To prove the* principle of virtual velocities *for forces acting in one plane on a point, and on a rigid body at different points.*

DEFINITION. If any forces as P_1 and P_2 act at a point as A in the figure, and this point is displaced through an indefinitely small space Aa, and we draw perpendiculars p_1a and p_2a from a upon the directions of the forces, then the distances Ap_1 and Ap_2 are called the virtual velocities of the forces P_1 and P_2; and Ap_1 being measured in the direction of force P_1 is called *positive*, Ap_2 being measured in the direction of force P_2 *produced* is called *negative.*

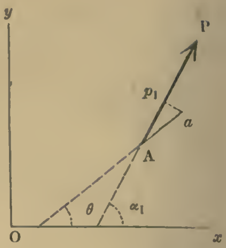


The principle of virtual velocities is thus enunciated: *If any number of forces be in equilibrium at one or more points of a rigid body, then if this body receive an indefinitely small disturbance, the* algebraic sum *of the products of each force into its virtual velocity is equal to zero.*

This principle is true when the forces in equilibrium act at any points and in any planes on a rigid body; but we shall in this treatise only prove the case when the forces act either at one point or at different points, in one plane, because the general case requires a knowledge of analytical geometry of three dimensions.

First. To prove the principle when the forces act all at one point.

Let A be the point at which the forces P_1, P_2, &c. . . . P_n, act.



Let α_1, α_2, α_3, &c. . . . α^n, be the angles they make respectively with Ox.

Let θ be the angle which the displacement Aa makes with Ox.

Let v_1, v_2, v_3, &c. . . . v_n, be the virtual velocities of the forces respectively; then in figure, $v_1 = Ap_1 = Aa \cos. aAp_1$

$$= Aa \cos. (\alpha_1 - \theta)$$

$$= Aa (\cos. \alpha_1 \cos. \theta + \sin. \alpha_1 \sin. \theta)$$

$$\text{and } P_1 . v_1 = P_1 Aa (\cos. \alpha_1 \cos. \theta + \sin. \alpha_1 \sin. \theta)$$

$$= Aa (\cos. \theta . P_1 \cos. \alpha_1 + \sin. \theta . P_1 \sin. \alpha_1)$$

Similarly, for P_2 we have

$$P_2 . v_2 = Aa (\cos. \theta . P_2 \cos. \alpha_2 + \sin. \theta . P_2 \sin. \alpha_2)$$

and so for the other forces, therefore, we have

$$P_1 . v_1 + P_2 . v_2 + P_3 . v_3 + \&c. \; . \; . \; . \; P_n . v_n = \Sigma (P . v) \text{; say,}$$

$$= Aa \{ \cos. \theta (P_1 \cos. \alpha_1 + P_2 \cos. \alpha_2 + \&c. \; . \; . \; . + P_n \cos. \alpha_n)$$

$$+ \sin. \theta (P_1 \sin. \alpha_1 + P_2 \sin. \alpha_2 + \&c. \; . \; . \; . + P_n \sin. \alpha_n) \}$$

But when there is equilibrium at a point

$$P_1 \cos. \alpha_1 + P_2 \cos. \alpha_2 + \&c. \; . \; . \; . + P_n \cos. \alpha_n = 0$$

$$P_1 \sin. \alpha_1 + P_2 \sin. \alpha_2 + \&c. \; . \; . \; . + P_n \sin. \alpha_n = 0$$

∴ we have $\Sigma (P . v) = 0$; or, the principle is true when the forces act all at one point.

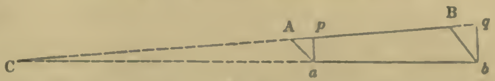
Second. Let the forces act at different points or particles of the body in one plane. We have now to consider these points connected together by rigid lines or rods without weight, which transmit the reactions of the particles upon each other. These reactions must be considered together with the other forces.

Let A_1, A_2, A_3, &c. . . . A_m, be the particles.

Let $r_{a_1 a_2}$ be the reaction of the particle $A_{\cancel{1}\,2}$ upon the particle $A_{\cancel{2}\,1}$
$r_{a_2 a_1}$ - - - $A_{\cancel{2}\,1}$ - - $A_{\cancel{1}\,2}$
$r_{a_1 a_3}$ - - - $A_{\cancel{1}\,3}$ - - $A_{\cancel{3}\,1}$
$r_{a_3 a_1}$ - - - $A_{\cancel{3}\,1}$ - - $A_{\cancel{1}\,3}$
&c. &c. &c.

Let $v_{a_1 a_2}$, $v_{a_2 a_1}$, $v_{a_1 a_3}$, $v_{a_3 a_1}$, &c. &c. be the corresponding virtual velocities;

then $r_{a_1a_2}=r_{a_2a_1}$, $r_{a_1a_3}=r_{a_3a_1}$, &c. &c. from the nature of reactions. Also $v_{a_1a_2}=-v_{a_2a_1}$, $v_{a_1a_3}=-v_{a_3a_1}$, &c., which we must shew.



Let A and B be the particles displaced to a and b. Draw the perpendiculars ap, bq. Then if the line ab is parallel to AB, the point to be proved is evidently true. When ab is not parallel to AB, let them meet when produced, if necessary, in some point C. Since the displacements are indefinitely small, the perpendiculars ap, bq coincide with circular arcs having C for center, and

$$Ca=Cp \qquad Cb=Cq$$

but $Ap=Cp-CA=Ca-CA$

$$Bq=Cb-CB=(Ca+ab)-(CA+AB)$$

$=Ca-CA$ . . . since $AB=ab$

$=Ap$, but measured in the opposite direction to the reaction of B upon A, and is therefore negative.

Let the sum of the products of all the *external* forces into their virtual velocities, acting on particle A_1 be $\Sigma(P_{a_1}.v_{a_1})$

those on - - A_2 be $\Sigma(P_{a_2}.v_{a_2})$

those on - - A_3 be $\Sigma(P_{a_3}.v_{a_3})$

&c. &c.

those on - - A_m be $\Sigma(P_{a_m}.v_{a_m})$

Since each particle is in equilibrium from the action of the forces upon it, we have from the first case,

$$0=\Sigma(P_{a_1}.v_{a_1})+r_{a_1a_2}.v_{a_1a_2}+r_{a_1a_3}.v_{a_1a_3}+\&c.$$
$$0=\Sigma(P_{a_2}.v_{a_2})+r_{a_2a_1}.v_{a_2a_1}+r_{a_2a_3}.v_{a_2a_3}+\&c.$$
$$0=\Sigma(P_{a_3}.v_{a_3})+r_{a_3a_1}.v_{a_3a_1}+r_{a_3a_2}.v_{a_3a_2}+\&c.$$
$$\&c. \qquad\qquad \&c.$$
$$0=\Sigma(P_{a_m}.v_{a_m})+r_{a_ma_1}.v_{a_ma_1}+r_{a_ma_2}.v_{a_ma_2}+\&c.$$

In taking the sum of the products for all the particles, the products of the reactions into their virtual velocities will disappear, being in pairs equal in magnitude with contrary signs; therefore we have,

$$\Sigma(P_{a_1}.v_{a_1})+\Sigma(P_{a_2}.v_{a_2})+\Sigma(P_{a_3}.v_{a_3})+\&c. \;.\;.\;.\; +\Sigma(P_{a_m}.v_{a_m})=0$$

or generally, when there is equilibrium,

$$\Sigma(P.v)=0.$$

27. CONVERSELY. *If the sum of the products of the forces into their virtual velocities be equal to zero, or* $\Sigma(P.v)=0$, *then there will be equilibrium.*

For if the forces be not in equilibrium, they will be equivalent either to a single force or a single couple. (Art. 24.)

In the first case, let R be the single resultant force, then a force equal and opposite to R will reduce the system to equilibrium; let u be its virtual velocity for any displacement. Since there is now equilibrium, we have, by the preceding article,

$$\Sigma(P.v)+R.u=0$$

But by hypothesis $\Sigma(P.v)=0$ $\therefore$ $R.u=0$; which being true for all small displacements of the body, we must have $R=0$, or the body was in equilibrium from the action of the original forces.

In the second case, if the forces were equivalent to a resultant couple, it would be balanced by an equal and opposite couple. Let the forces of this opposite couple be Q and Q', and their virtual velocities for any displacement be q and q' respectively. Since they will reduce the system to equilibrium, we have by the preceding article

$$\Sigma(P.v)+Q.q+Q'.q'=0$$

but $\Sigma(P.v)=0$ $\therefore$ $Q.q+Q'.q'=0$ for all displacements, which is impossible unless Q and Q' each $=0$, since they are equal and parallel forces, and act at different points.

CHAPTER V.

ON THE CENTER OF GRAVITY.

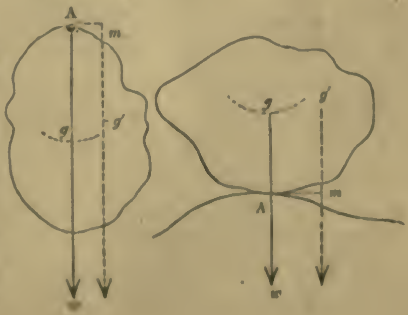
28. *The center of gravity of a body is that point at which the whole weight of a body may be considered to act, and would produce the same mechanical effect as the weight of the body actually does.*

The weights of all the particles of a body, acting vertically downwards, are parallel forces, so that the center of gravity coincides with the center of parallel forces for such weights.

From the definition it arises, that if the center of gravity of a body be a fixed point, the body will balance about that point in all positions. This property of the center of gravity often furnishes the means of determining its position practically. In regular and symmetrical figures, as cubes, spheres, cylinders, thin plates which are circular, elliptic, or regular polygons, &c. it is evidently the center of the body, or point about which it is symmetrical.

29. Prop. *If a body be in equilibrium, suspended from any point, or resting with one point of contact upon another body, then the center of gravity lies in the vertical line through that point of suspension or contact respectively.*

Let A in the figures be the points of suspension and contact respectively; draw the vertical lines Aw. If the whole weight of the body act in these vertical lines, it will be supported by the reactions of the fixed points A, or when the centers of gra-



vity g are in these lines. If the centers of gravity were not in these lines, but at some points as g'; drawing the vertical lines $g'm$ through g', and the horizontal lines Am, then w, the weight of the body acting at g', would have a moment $w \times Am$, which being unbalanced, the body could not be in equilibrium, contrary to the supposition.

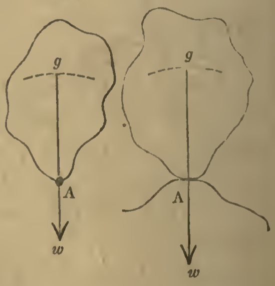
Definitions. *A body is said to rest in* stable equilibrium *when, after receiving a slight disturbance, it returns to its first position.*

It is said to rest in unstable equilibrium *when, after receiving a slight disturbance, it moves from its position of equilibrium.*

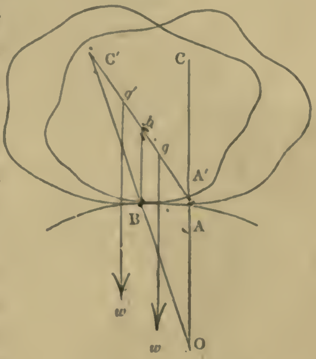
It is said to rest in neutral equilibrium *when, after being disturbed slightly, it still rests in equilibrium.*

30. Prop. *When a body rests in stable equilibrium, its center of gravity is in the lowest position it can take; and when in unstable equilibrium, it is in the highest position it can take.*

In the first figure of the last article the body rests in stable equilibrium, and the center of gravity g would, on disturbance, describe a circular arc about the point of suspension A, and therefore would rise on the body being disturbed. In the second figure also, if the equilibrium be stable, the center of gravity will rise on disturbance, from the change of the point of contact from A to a neighbouring point. In the annexed figures, whilst the vertical lines through the centers of gravity g pass through the points of suspension or contact A, the body will rest in equilibrium; but on a small disturbance being given to the bodies, the centers of gravity, falling out of the vertical lines through the points of suspension and contact, will come to a lower position than at first, and the weight will have a moment turning the body further from its position of equilibrium, which therefore in this case is unstable.



31. Prop. *To find the conditions that the equilibrium may be stable, unstable, or neutral, when the spherical surface of a body rests upon another spherical surface.*



Let A be the point of contact of the spherical surfaces, C the center of the upper surface, O that of the lower. Let $CA = r$, $OA = r'$. Let the body receive a small disturbance, so that the point C comes to C' in the plane of the figure, and the point of contact is now B, A' being the new position of A. Join O and C', then $OC' = r + r'$. Draw Bb, a vertical line meeting $C'A'$ in b. Then if the center of gravity of the body falls between A' and b, as at g in the figure, the equilibrium will be stable, for the moment of the weight (w) of the body will bring the body back to its first position. If the center of gravity falls beyond b from A', as at g', the vertical line through g' will fall beyond B, and the moment of the weight will cause the body to move further from its first position, and the equilibrium will be unstable.

If the vertical line through the center of gravity passes through b, the body will be still in equilibrium, which is therefore neutral.

When the displacement is indefinitely small, A' will be indefinitely near A, and, by similar triangles, we have

$$A'b : AC' :: OB : OC'$$

$$\text{or, } A'b : r :: r' : r + r'$$

$$\text{or, } A'b = \frac{r r'}{r + r'}$$

consequently, when the equilibrium is

$$\text{stable, } Ag \text{ is less than } \frac{r r'}{r + r'}$$

$$\text{unstable, } Ag' \text{ is greater than } \frac{r r'}{r + r'}$$

$$\text{neutral, } Ag \text{ is equal to } \frac{r r'}{r + r'}$$

When the lower surface is a horizontal plane, the radius of the body at the point of contact is always vertical, and the equilibrium is

stable, when Ag is less than r;
unstable, when Ag' is greater than r;
neutral, when Ag is equal to r.

32. Prop. *To find the condition that a body placed on a plane surface may stand or fall.*

fig. 1.

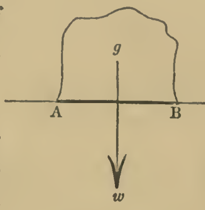


fig. 2.

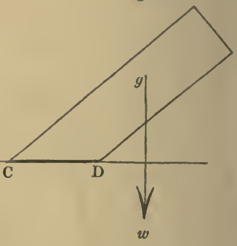


fig. 3.

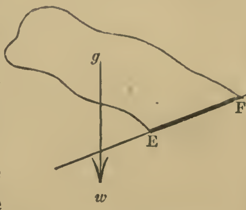
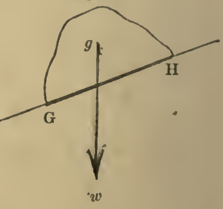


fig. 4.



Let figures 1 and 2 represent the sections of bodies by vertical planes through their centers of gravity (g), which rest on a horizontal plane.

Let figures 3 and 4 represent, similarly, bodies resting on an inclined plane, down which they are prevented from slipping by friction.

Drawing vertical lines through the centers of gravity of the bodies, they will be the directions in which the weight of each acts, as $g\ w$ in the figures.

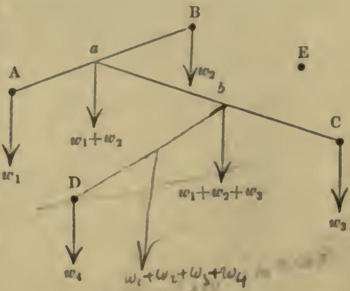
Now, in figures 1 and 4 the weight cannot cause the body to turn about either A or B in figure 1, or G or H in figure 4, because its effect is destroyed by the resistance of the plane. But in figures 2 and 3 the weight will have a moment about D in figure 2, and about E in figure 3, which is not neutralised by the resistance of the plane, and the bodies consequently will fall over.

The condition, therefore, that a body placed on a plane shall stand or fall is, that the vertical line through its center of gravity falls within or without the base, respectively.

TO FIND THE POSITION OF THE CENTER OF GRAVITY IN SYSTEMS OF PARTICLES AND IN RIGID BODIES.

33. Prop. *To find the center of gravity of any number of heavy particles whose places are given.*

Let *A*, *B*, *C*, *D*, *E*, &c. be the particles whose weights w_1, w_2, w_3, &c. act in the vertical direction indicated by the arrows from *A*, *B*, *C*, &c. and therefore constitute a system of parallel forces. w_1 and w_2 will have a resultant $= w_1 + w_2$ acting at a point *a*, such that $w_1 \times Aa = w_2 \times Ba$. (Art. 12.)



Compounding the weight $w_1 + w_2$ at *a* with another weight w_3 acting at *C*, they will have a resultant $= w_1 + w_2 + w_3$ acting at a point *b*, such that

$$(w_1 + w_2) \times ab = w_3 \times Cb$$

Compounding the weight $w_1 + w_2 + w_3$ acting at *b*, with another weight w_4 acting at *D*, we should find the point at which the resultant $w_1 + w_2 + w_3 + w_4$ acted, and so onwards for any number of particles whose positions were given.

The position of the point at which the final resultant weight acts is thus determined, and is the same point whatever be the order in which we compound the weights; so that a system of particles or a rigid body can never have more than one center of gravity.

The positions of the points *a*, *b*, &c. depend on the weights w_1, w_2, w_3, &c. and on the positions of the points *A*, *B*, *C*, &c. with respect to each other, but not at all on the directions of the arrows with regard to the lines *AB*, *aC*, &c.; so that gravity acting vertically downwards, we may turn the whole system into any new position, the weights and relative positions of the particles remaining the same, and shall find the center of gravity in the same point as before.

Cor. If the center of gravity of a system of particles rigidly connected, or of a rigid body, be supported, the whole system will be supported, and the system or body will balance about the center of gravity in all positions.

34. If the positions of the particles be given with respect to a fixed origin and co-ordinate axes, we should find the position of the center of gravity by the same process as in article 16; the weights of the particles being the forces, and the whole weight of the body being the resultant, or the sum of the weights of all the particles.

If $m =$ the mass of *any* particle, and therefore proportional to its weight, x, y its co-ordinates, and $\bar{x}$, $\bar{y}$ the co-ordinates of the center of gravity of the system, we have, by article 16,

$$\bar{x} = \frac{\Sigma(m\,.\,x)}{\Sigma(m)} \quad . \;\; . \;\; . \quad \text{or} = \frac{m_1x_1+m_2x_2+m_3x_3+\&\text{c.}}{m_1+m_2+m_3+\&\text{c.}}$$

$$\bar{y} = \frac{\Sigma(m\,.\,y)}{\Sigma(m)} \quad . \;\; . \;\; . \quad \text{or} = \frac{m_1y_1+m_2y_2+m_3y_3+\&\text{c.}}{m_1+m_2+m_3+\&\text{c.}}$$

where x_1, y_1 are the co-ordinates of the particle m_1; x_2, y_2 those of m_2, &c. &c. If the particles are all in the axis of x, $y_1=0$, $y_2=0$, &c. &c. and $\bar{y}=0$.

35. The formulæ of the last article are applicable to the solution of problems where the centers of gravity of the parts are given to find the center of gravity of the whole body, and where the centers of gravity of the whole body and some of the parts are given, to find the center of gravity of the remaining part; for the weight of each part being taken at its center of gravity, we treat the problem as if a heavy particle of that weight were placed there. Thus, let M equal the whole mass of a body; $\bar{x}$, $\bar{y}$ the co-ordinates of its center of gravity; M_1, M_2, $\bar{x}_1$, $\bar{y}_1$, $\bar{x}_2$, $\bar{y}_2$, the corresponding quantities for its two parts; we have

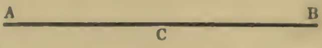
$$\bar{x} = \frac{M_1\bar{x}_1+M_2\bar{x}_2}{M} \qquad \bar{y} = \frac{M_1\bar{y}_1+M_2\bar{y}_2}{M}$$

If M, M_1, and $\bar{x}$, $\bar{y}$, $\bar{x}_1$, $\bar{y}_1$, were given, we have

$$M_2 = M - M_1$$

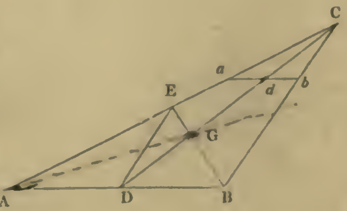
$$\bar{x}_2 = \frac{M\bar{x}-M_1\bar{x}_1}{M_2} \qquad \bar{y}_2 = \frac{M\bar{y}-M_1\bar{y}_1}{M_2}$$

36. Ex. 1. *To find the center of gravity of a uniform physical straight line.*



If AB be the uniform straight line, it will balance on a fulcrum or fixed point at C its middle point, which will be its center of gravity by art. 28; for we may consider the line as made up of a series of equal particles in pairs, at equal distances on opposite sides of C, and the weights of each pair would have their resultant weight acting at C, the middle point between them, or the resultant weight of the whole line would act at C; and this weight being supported by the reaction of the fulcrum, the line will be supported, and its center of gravity will be at C, its middle point.

Ex. 2. *To find the center of gravity of a triangular plate, of uniform thickness and density.*



Let ABC be the triangular plate of which the thickness is inconsiderable. Draw from C the line CD bisecting AB in D, and from B the line BE bisecting AC in E. Let G be the intersection of CD and BE, then G is the center of gravity of the triangle. For we may consider the triangle to be made up of physical lines, each parallel to AB; let adb be any one of these lines, meeting CD in d, which will be its middle point, because by similar triangles we have

$$\begin{aligned} Cd : da &:: CD : DA \\ &:: CD : DB \\ &:: Cd : db \end{aligned}$$

The line ab has therefore its center of gravity at d.

Similarly it is shewn that every line parallel to AB will be bisected by CD, and have its center of gravity in that line; therefore the center of gravity of the triangle will be in this line.

In the same way it is shewn that the center of gravity will be in the line BE: consequently it must be at the intersection of these lines, or at the point G in the figure.

Join the points D and E. The line DE is parallel to BC, because the sides AC, AB are cut proportionally in D and E, and $DE = \frac{1}{2} BC$.

Also the triangles EDG, BCG are similar.

$$\therefore ED : BC :: DG : CG$$

$$:: 1 : 2$$

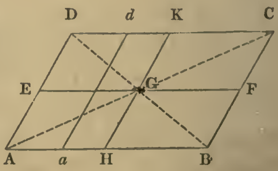
$$\text{or, } DG = \tfrac{1}{2} . CG = \tfrac{1}{3} . CD$$

$$\text{So also, } EG = \tfrac{1}{3} . BE$$

Therefore to find the center of gravity of a triangular plate, we draw a line from any angle to the middle of the opposite side, and measure along this line $\frac{2}{3}$ its length from the angle, or $\frac{1}{3}$ from the bisection of the side, and the point so found is the center of gravity required.

Ex. 3. *To find the center of gravity of a parallelogram of which the density is the same at every point, and the thickness uniform but very small.*

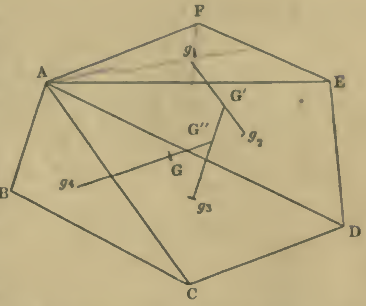
Let $ABCD$ be the parallelogram; bisect the sides AD, BC in E and F, and join EF; also bisect AB and CD in H and K, and join HK; let G be the intersection of EF and HK, then G is the center of gravity of the parallelogram.



For the parallelogram may be considered made up of physical lines as ad parallel to AD; each of these will be bisected by the line EF, and therefore the center of gravity of the parallelogram will be in this line. Similarly, the center of gravity will be in the line HK, and is therefore the point G at the intersection of these lines.

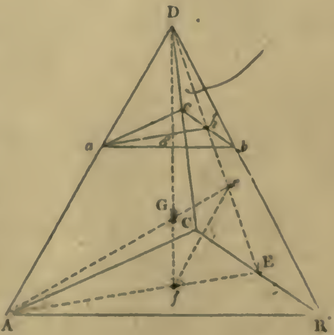
It is also evidently the intersection of the diagonals of the parallelogram.

Ex. 4. *To find the center of gravity of a polygonal plate, of uniform density and thickness.*



Let $ABCDEF$ be the polygon. Draw the lines AE, AD, AC, dividing it into triangles. When the polygon is given, these triangles will be known, and their centers of gravity will be found by Ex. 2. Let g_1, g_2, g_3, g_4, be these centers of gravity respectively. We may consider the mass of each triangle as a heavy particle at its center of gravity. Compounding the masses at g_1 and g_2, by art. 33, their center of gravity will be at some point as G' in the line joining g_1 and g_2. Compounding the mass of the two triangles at G' with the mass of the next triangle at g_3, we shall have the center of gravity of the three triangles at some point as G'', and so onwards; the last point so found will be the center of gravity G of the whole figure.

Ex. 5. *To find the center of gravity of a triangular pyramid, of uniform density.*



Let $ABCD$ be the triangular pyramid. Bisect the edge BC in E, and draw the lines AE, DE; in AE take $Af = \frac{2}{3}AE$, in DE take $De = \frac{2}{3}DE$, then e and f are the centers of gravity of the triangular faces of the pyramid DCB, ABC respectively. Join Df, Ae; these lines intersect in a point G, which is the center of gravity of the pyramid.

For we may consider the pyramid made up of triangular plates parallel to any one of its faces. Let abc be such a plate parallel to the face ABC. The parallel planes meet the plane DCB in cb, CB, therefore these lines are parallel, and cb is bisected by the line DhE in h; for

$$\begin{aligned} Dh : ch &:: DE : CE \\ &:: DE : EB \\ &:: Dh : hb \end{aligned}$$

therefore $ch = hb$.

Similarly, the lines ah and AE are parallel, and Df, a line in the plane of the triangle AED, cuts them proportionally. Let g be the intersection of Df and ah,

$$Dg : ag :: Df : Af$$

$$\text{and } gh : Dg :: fE : Df$$

$$\begin{aligned} \text{Compounding, } gh : ag &:: fE : Af \\ &:: 1 : 2 \end{aligned}$$

therefore g is the center of gravity of the triangle abc. Similarly it may be shewn that the center of gravity of every section parallel to ABC is in the line Df. In the same way it may be shewn that the center of gravity of every section parallel to the face BCD is in the line Ae. The center of gravity of the whole pyramid must therefore be in each of these lines, which are both in the plane AED. Let G be the intersection of Ae and Df, or the center of gravity of the pyramid; join fe. Since AE, DE are cut proportionally in e and f, ef is parallel to AD, and

$$\begin{aligned} ef : AD &:: fE : AE \\ &:: 1 : 3 \end{aligned}$$

Also the triangles AGD, fGe are similar, and

$$\begin{aligned} fe : AD &:: fG : GD \\ &:: 1 : 3 \end{aligned}$$

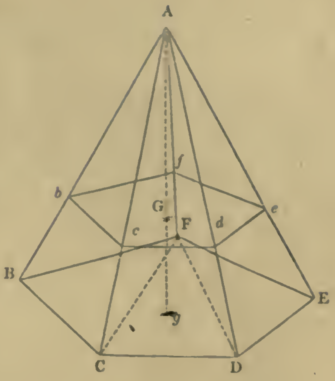
$$\text{or, } fG = \tfrac{1}{3} GD = \tfrac{1}{4} Df, \text{ and } DG = \tfrac{3}{4} Df$$

$$\text{Similarly } eG = \tfrac{1}{4} Ae, \text{ and } AG = \tfrac{3}{4} Ae$$

Or, to find the center of gravity of a triangular pyramid, we must draw a line from any one of the solid angles to the center of gravity of the opposite face, and measure $\frac{3}{4}$ of that line from the angle for the point required.

Ex. 6. *To find the center of gravity of a pyramid whose base is any polygon.*

Let $BCDEF$ be the polygon which is the base of the pyramid whose vertex is A. Joining CF and DF, we divide the



polygon into triangles, and planes passing through *CF*, *DF*, and the vertex *A*, will divide the pyramid into triangular pyramids. Drawing a line from *A* to the center of gravity of any one of the triangular bases, and measuring $\frac{3}{4}$ of that line from *A*, we shall have the center of gravity of the triangular pyramid on that base. If we take a plane *bcdef* through this point parallel to the base, it will cut all lines drawn from *A* to the polygonal base in the same proportion, and therefore the centers of gravity of all the triangular pyramids will be in this plane, and consequently the center of gravity of the whole pyramid also, because the mass of each pyramid may be considered a heavy particle at its center of gravity.

Again. If *g* be the center of gravity of the polygon, and we join *Ag*, it can be shewn that the center of gravity of every section parallel to the base will be in the line *Ag*, and therefore the center of gravity of the whole pyramid will be in this line; and since it is also in the plane *bcdef*, it is at the point *G* where *Ag* meets the plane.

Hence, to find the center of gravity of any pyramid on a polygonal base, we must draw a line from the vertex to the center of gravity of the polygon, and measure $\frac{3}{4}$ of it from the vertex, or $\frac{1}{4}$ from the base.

COR. The above rule holds good whatever may be the number of sides of the polygon, and is therefore true when the number becomes indefinitely great, or when the base becomes a continued closed curve, as a circle, an ellipse, oval, &c. Or, the center of gravity of a cone, right or oblique, and on any base, is found by drawing a line from the vertex of the cone to the center of gravity of the base, and measuring $\frac{3}{4}$ of the line from the vertex, or $\frac{1}{4}$ from the base.

Ex. 7. *To find the center of gravity of a frustum of a cone or pyramid cut off by a plane parallel to the base.*

Let the length of the line drawn from the vertex of the cone, when complete, to the center of gravity of the base $= a$. Let the length of the same line to where it meets the smaller end of the frustum $= a'$. Using the formula of article 35, we have

$$\bar{x}_2 = \frac{M\bar{x} - M_1\bar{x}_1}{M_2}$$

where $\bar{x} = \frac{3}{4}a$, $\bar{x}_1 = \frac{3}{4}a'$.

Also. Similar solids have their volumes proportional to the cubes of their lines similarly situated, and the part of the cone or pyramid cut off by a plane parallel to the base is similar to the whole cone or pyramid, therefore we have $\frac{M_1}{M} = \frac{a'^3}{a^3}$,

$$\text{and } M_2 = M - M_1 = M\left(1 - \frac{a'^3}{a^3}\right)$$

$$\text{and } \bar{x}_2 = \frac{M\frac{3}{4}a - M\left(\frac{a'^3}{a^3}\right)\frac{3}{4}a'}{M\left(1 - \frac{a'^3}{a^3}\right)}$$

$$= \frac{3}{4}\frac{a^4 - a'^4}{a^3 - a'^3}$$

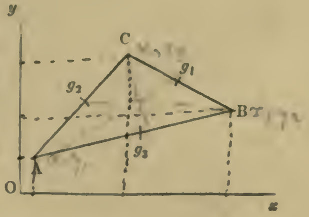
$$= \frac{3}{4}\frac{(a + a')(a^2 + a'^2)}{a^2 + aa' + a'^2}$$

which gives the distance of the center of gravity from the vertex of the cone or pyramid; and the distance from the center of gravity of the base along the same line is

$$a - \frac{3}{4}\frac{(a + a')(a^2 + a'^2)}{a^2 + aa' + a'^2} = \frac{a}{4} - \frac{3a'^3}{4(a^2 + aa' + a'^2)}$$

Ex. 8. *To find the center of gravity of the perimeter of a given triangle in terms of the co-ordinates of its angular points.*

We suppose the perimeter of the triangle to be three uniform physical lines whose weights are proportional to their lengths. Let a, b, c be the sides respectively opposite to the angles A, B, C. The centers of gravity will be each at the middle point of the side, as at g_1, g_2, g_3, in the figure.



Let $x_1 y_1$ be co-ordinates of A to origin O
$x_2 y_2$ . . . B
$x_3 y_3$ . . . C

then the co-ordinates of g_1 are $\frac{1}{2}(x_2+x_3)$, $\frac{1}{2}(y_2+y_3)$
g_2 are $\frac{1}{2}(x_1+x_3)$, $\frac{1}{2}(y_1+y_3)$
g_3 are $\frac{1}{2}(x_1+x_2)$, $\frac{1}{2}(y_1+y_2)$

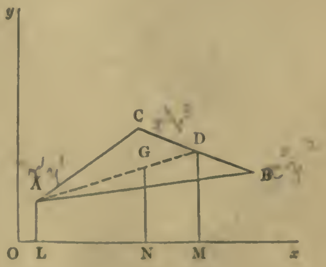
and $\bar{x}, \bar{y}$ being the co-ordinates required, the formulæ of Article 34 give us

$$\bar{x} = \frac{a(x_2+x_3)+b(x_1+x_3)+c(x_1+x_2)}{2(a+b+c)}$$

$$\bar{y} = \frac{a(y_2+y_3)+b(y_1+y_3)+c(y_1+y_2)}{2(a+b+c)}$$

Ex. 9. *To find the co-ordinates of the center of gravity of a triangular plate.*

Let x_1y_1, x_2y_2, x_3y_3 be the co-ordinates of the points A, B, C respectively.

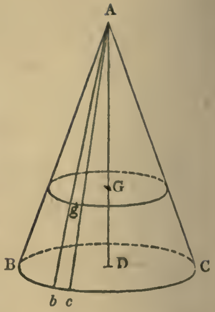


Let the line AD bisect BC in D; the center of gravity being G, we have $AG = \frac{2}{3}AD$. The co-ordinates of D are $\frac{1}{2}(x_2+x_3)$, $\frac{1}{2}(y_2+y_3)$; and if $\bar{x} = ON$, $\bar{y} = GN$, be the co-ordinates of G, we have

$$ON = OL + \tfrac{2}{3}(OM - OL)$$
$$GN = AL + \tfrac{2}{3}(DM - AL);$$
$$\text{or, } \bar{x} = x_1 + \tfrac{2}{3}\left\{\tfrac{1}{2}(x_2+x_3) - x_1\right\}$$
$$= \tfrac{1}{3}(x_1+x_2+x_3)$$
$$\text{Similarly, } \bar{y} = \tfrac{1}{3}(y_1+y_2+y_3)$$

Ex. 10. *To find the center of gravity of the surface of a right cone.*

We consider the surface of the cone as a sheet of matter equally dense at every point; and as it is symmetrical with respect to the axis of the cone, its center of gravity must be in that line.

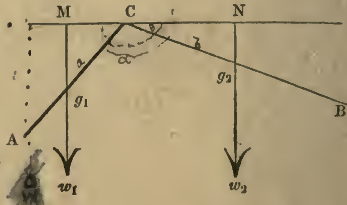


Again. If we draw the straight lines Ab, Ac, from the vertex to the circumference of the base, so that bc is an indefinitely small arc, the center of gravity (g) of the triangle Abc is at a distance Ag from $A = \frac{2}{3}Ab$. This is true for every such elementary triangle which can be formed on the surface of the cone; or their centers of gravity are in a circle whose center is G in the axis AD of the cone, such that $AG = \frac{2}{3}AD$; and G is the center of gravity of the surface of the cone required.

EXAMPLES ON THE PRECEDING CHAPTERS.

Ex. 1. Two beams, connected together at a given angle, turn about a horizontal axis at their point of meeting; find the position of equilibrium which they will take by the action of their own weights.

Let AC, BC be the beams suspended from C, and inclined to each other at an angle α. Since C is a fixed point, the only condition of equilibrium is, that the moments of the weights about C may balance.



Let g_1, g_2 be centers of gravity of the beams, and $g_1C = a$, $g_2C = b$. Let w_1 = weight of beam AC acting at g_1; w_2 that of BC acting at g_2. Draw MCN, a horizontal line, meeting the vertical lines in which the weights of the beams act in M and N.

In equilibrium we have $w_1 \times CM = w_2 \times CN$. Let θ = angle BCN, which is to be found; we have

$$w_1 \times Cg_1 \cos. ACM = w_2 \times Cg_2 \cos. BCN$$

$$\text{or, } w_1 a \cos. (180 - \overline{\alpha + \theta}) = w_2 b \cos. \theta$$

$$\text{whence } \tan. \theta = \frac{w_2 b + w_1 a \cos. \alpha}{w_1 a \sin. \alpha}$$

which gives the position of the beam as required.

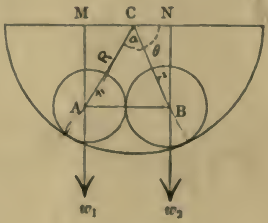
Ex. 2. When a given weight (W) is hung from the end of one of the beams (A), as in the last question, shew that

$$\tan. \theta = \frac{w_2 b + (W.AC + w_1 a) \cos. \alpha}{(W.AC + w_1 a) \sin. \alpha}$$

Ex. 3. Two beams, as in Ex. 1, are suspended from (B) one end: shew that if θ be the angle which the upper one makes with a horizontal line, we have in equilibrium,

$$\tan. \theta = \frac{(w_1 + w_2) BC - (w_2 b + w_1 a \cos. \alpha)}{w_1 a \sin. \alpha}$$

Ex. 4. Two spheres of unequal radii, but of the same material, are placed in a hemispherical bowl; find the position they take when in equilibrium.



N.B.—This, and similar problems of bodies resting in equilibrium in a hemispherical bowl, can be reduced to problems like the preceding. For if the center of the hemisphere C in the figure were a fixed point, and connected by rigid rods AC, BC, without weight, to the centers of the spheres, we might suppose the hemisphere removed without changing the conditions of equilibrium.

Let w_1 and w_2 be the weights of the spheres A and B, whose radii are r_1 and r_2 respectively. Let R be the radius of the bowl. Then if the angle $ACB = \alpha$ in triangle ABC, we have $AB = r_1 + r_2$, $AC = R - r_1$, $BC = R - r_2$, and

$$\cos. \alpha = \frac{AC^2 + BC^2 - AB^2}{2AC.BC}$$

$$= \frac{(R - r_1)^2 + (R - r_2)^2 - (r_1 + r_2)^2}{2(R - r_1)(R - r_2)}, \text{ which gives } \alpha.$$

The position of the spheres will be known if the angle BCN be known; let it $= \theta$. The weights are proportional to the volumes of the spheres, or to the cubes of the radii;

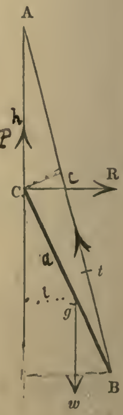
$$\text{or, } \frac{w_1}{w_2} = \frac{r_1^3}{r_2^3}, \text{ and } w_1 \times CM = w_2 \times CN$$

$$\text{or, } w_1(R-r_1)\cos.(180-\overline{\alpha+\theta}) = w_2(R-r_2)\cos.\theta$$

$$\text{whence } \tan.\theta = \frac{r_2^3(R-r_2)+r_1^3(R-r_1)\cos.\alpha}{r_1^3(R-r_1)\sin.\alpha}$$

Ex. 5. A heavy beam has one end resting against a smooth wall, and the other tied to a cord which is fastened at a point directly above the point where the beam rests; find the forces which keep the beam in equilibrium.

Let CB be the beam in the figure, AB the cord; A and C being points on the wall. The weight of the beam (w), the distance ($Cg = a$) of the center of gravity from the end against the wall, the length of the beam (l), the length of the cord (c), and the distance (h) of the points A and C must be given. The angles A, B, C, will be known.



Let $t =$ the tension in the cord.

The beam will press at C against the wall, and we may resolve this pressure into a vertical and horizontal part; the latter, perpendicular to the wall, will be destroyed by its reaction (R); but since the wall is smooth, the vertical component can be balanced only by an opposite force P.

If we take into account all the forces which act on the beam, we may treat it as a free body in equilibrium from their action, and apply the conditions of equilibrium investigated in Chapter IV., namely:

$$\Sigma(X) = 0$$

$$\Sigma(Y) = 0$$

$$\Sigma(Xy - Yx) = 0$$

Therefore, resolving the forces vertically and horizontally, we have

$$P + t\cos. A - w = 0 \qquad (1)$$

$$R - t\sin. A = 0 \qquad (2)$$

Since the origin of co-ordinates may be taken any where, we may fix it at a point where we avoid moments of the unknown forces; therefore, fixing it at *C*, we have, for the equation of moments,

$$w \times \sin. C \times Cg - t \times \sin. B \times CB = 0$$

$$\text{or, } t = w\frac{a.\sin. C}{l.\sin. B}$$

$$= w\frac{a}{l}\cdot\frac{c}{h}$$

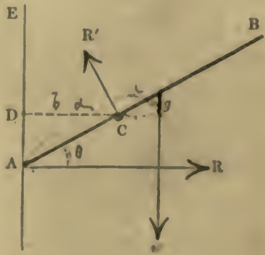
which gives the tension in the cord.

Substituting in the equations (1) and (2), we have

$R = w\dfrac{ac}{lh}\sin. A$, which gives the pressure against the wall.

$P = w\left(1 - \dfrac{ac}{lh}\cos. A\right)$, which gives *P*, as required.

Ex. 6. A heavy beam rests upon a peg, with one end against a smooth vertical wall: find the position of equilibrium.



Let *ACB* be the beam, resting at *A* against the vertical wall *ADE*, and upon the peg *C*.

The center of gravity, when there is equilibrium, will be evidently at some point, as *g*, beyond *C* from *A*.

Let w = the weight of the beam; R = reaction of the wall perpendicular to itself at *A*; R' = reaction of the peg perpendicular to the beam at *C*. These three forces keep the beam in equilibrium when making some angle θ with the horizontal direction, which is to be found.

Let $Ag = a$, and $CD = b$ = perpendicular distance of the peg from the wall, which must be given.

Resolving the forces horizontally and vertically, we have

$$R - R' \sin.\theta = 0 \tag{1}$$

$$w - R' \cos.\theta = 0 \tag{2}$$

Taking the moments about C, we have

$$w \,.\, Cg \cos.\theta - R \,.\, CD \,.\tan.\theta = 0 \tag{3}$$

$$\text{or, } w(a - b \text{ secant } \theta) \cos.\theta - Rb \,.\tan.\theta = 0$$

Multiply (1) by cos. θ, (2) by sin. θ, and subtract, and we have

$$R \cos.\theta - w \sin.\theta = 0$$

$$\therefore\ R = w \tan.\theta$$

$$\text{Substituting, } w(a - b \sec.\theta) \cos.\theta - wb \tan.^2\theta = 0$$

$$\text{or, } a \cos.\theta - b\,(1 + \tan.^2\theta) = 0$$

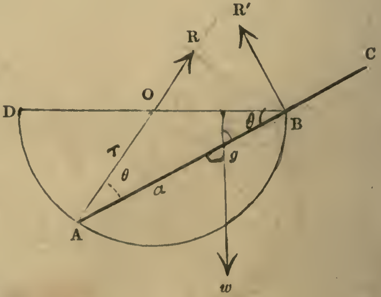
$$\text{whence, } \cos.\theta = \sqrt[3]{\frac{b}{a}}, \text{ and } b \text{ must be less than } a.$$

Ex. 7. Prove that in the last question the same result is obtained if we resolve parallel and perpendicular to the beam, and take the moments about either A or g in place of C.

Ex. 8. A heavy beam lies partly in a smooth hemispherical bowl and partly over one edge: find the position of equilibrium.

Let ABC be the beam, resting on the surface of the hemispherical bowl at A, and on the edge at B.

Let O be the center of the bowl DAB, whose radius $= r$, and BOD horizontal.



The center of gravity of the beam will be at some point g within the bowl. Let $Ag = a$.

The beam is supported by the reaction of the bowl at A perpendicular to the surface, or in AO, let it$=R$; by the reaction of the edge at B perpendicular to the beam, let it$=R'$; and by the weight of the beam (w) acting at g.

Let θ=angle ABO=angle BAO, which is to be found.

By the artifice of resolving in the direction of the beam and taking moments about B, we avoid expressions involving the unknown reaction R', and have, parallel to AB,

$$R \cos. \theta - w \sin. \theta = 0$$

$$\text{or, } R = w \tan. \theta$$

For moments about B,

$$R . AB \sin. \theta - w . Bg \cos. \theta = 0$$

$$\text{or, } R . 2r \cos. \theta . \sin. \theta - w (2r \cos. \theta - a) \cos. \theta = 0$$

Substituting for R, and omitting the common factors,

$$2r \tan. \theta . \sin. \theta - 2r \cos. \theta + a = 0$$

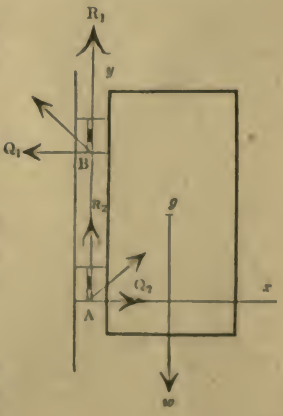
$$\text{whence } 2r - 4r \cos.^2 \theta + a \cos. \theta = 0$$

$$\text{or, } \cos. \theta = \frac{a \pm \sqrt{32 r^2 + a^2}}{8r}$$

in which the + sign only is admissible.

Ex. 9. Solve the last example by following the method of Article **24** in every step; taking A for the origin of co-ordinates, and AB for the axis of x.

Ex. 10. Find the horizontal strain on the hinges of a given door, and shew that the vertical pressures are indeterminate.



Let the figure annexed represent the door, of which the hinges are A and B.

Let g be the center of gravity at which the weight (w) of the door acts.

The door is in equilibrium from its weight w at g, and the reactions of the hinges represented by the oblique arrows at A and B.

Let A be the origin of co-ordinates as in the figure; Ax the axis of x; Ay

the axis of y; and let $x=a$, $y=b$ be the co-ordinates of g, $x=0$, $y=h$ those of the hinge B.

Let the resolved parts of the reactions at B be Q_1 horizontally, and R_1 vertically, and let Q_2, R_2 be those at A respectively, as in figure.

$$\text{Then, } \Sigma(X)=0=Q_2-Q_1 \qquad \text{or, } Q_1=Q_2 \qquad (1)$$

$$\Sigma(Y)=0=w-R_1-R_2 \qquad (2)$$

$$\Sigma(Xy-Yx)=0=w.a-Q_1.h \qquad (3)$$

From (3) and (1) we have

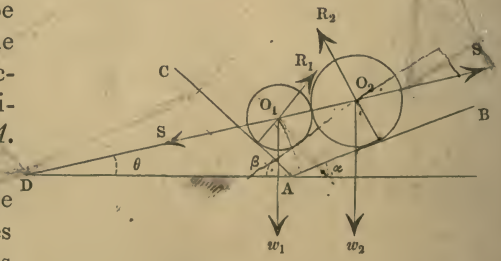
$$Q_1=\frac{wa}{h}=Q_2$$

which gives the horizontal strain; and it is the same at each hinge in magnitude, but opposite in direction.

Again. From (2) we have $\boldsymbol{R_1+R_2=w}$; but we have no other relation to enable us to determine the separate values of R_1 and R_2, which are therefore indeterminate.

Ex. 11. Two given smooth spheres rest in contact on two smooth planes, inclined at given angles to the horizon; to find their position of equilibrium.

Let AB, AC be the planes, making the angles α and β respectively with the horizontal line through A.



Let O_1, O_2 be the centers of the spheres at which their weights w_1 and w_2 respectively act.

Let R_1 and R_2 be the reactions of the planes at the points of contact, perpendicular to themselves, and therefore passing through the centers of the spheres to which they are tangents.

Let S equal the mutual pressure of the spheres at their point of contact, acting in the line passing through their centers; let this line $O_2\ O_1\ D$ make the angle θ with the horizontal line AD. It is required to find θ.

Each sphere is in equilibrium from its own weight, the reaction of the plane against which it rests, and the pressure of the other sphere. By the artifice of resolving in the directions of each plane for the equilibrium of each sphere, we avoid equations involving the unknown reactions R_1 and R_2, and have, in the direction of AB,

$$w_2 \sin.\alpha - S \cos.(\alpha-\theta) = 0 \qquad (1)$$

in direction of AC,

$$w_1 \sin.\beta - S \cos.(\beta+\theta) = 0 \qquad (2)$$

Eliminating S, we have

$$w_2 \sin.\alpha.\cos.(\beta+\theta) = w_1 \sin.\beta.\cos.(\alpha-\theta)$$

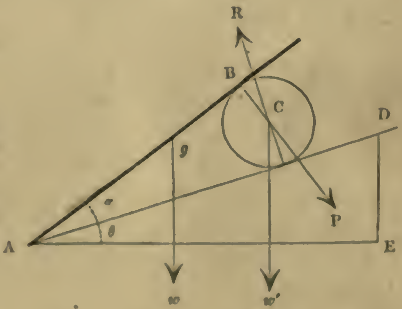
$$\text{whence } \tan.\theta = \frac{w_2 \sin.\alpha.\cos.\beta - w_1 \sin.\beta.\cos.\alpha}{(w_1+w_2)\sin.\alpha.\sin.\beta}$$

$$= \frac{w_2 \cot.\beta - w_1 \cot.\alpha}{w_1+w_2}$$

Ex. 12. A sphere is sustained upon an inclined plane by the pressure of a beam movable about the lowest point of the inclined plane; given the position of the beam, required that of the plane.

Let AgB be the beam, movable about A.

Let w = weight of the beam, acting at its center of gravity g; B the point of contact with the sphere, whose center is C; let w' = weight of the sphere.



The sphere is in equilibrium, from the reaction (R) of the plane at the point of contact, from the pressure (P) of the beam at B, and from its own weight; these three forces all act through the center C.

Let $Ag=a$, $AB=b$, angle BAD, which the beam makes with the plane, $=\alpha$, these are given; or, in place of either one of the two latter, we may have the radius of the sphere given.

Let the angle $DAE=\theta$ the elevation of the inclined plane, which is to be found, when there is equilibrium.

For the condition of equilibrium of the beam, taking moments about A,

$$P\times AB=w\,.\,Ag\cos.\,\overline{\alpha+\theta}$$

$$\text{or, } P=w\frac{a}{b}\cos.\,\overline{\alpha+\theta}$$

For the condition of equilibrium of the sphere, resolving the forces in the direction of AD, we have

$$w'\sin.\,\theta-P\sin.\,\alpha=0$$

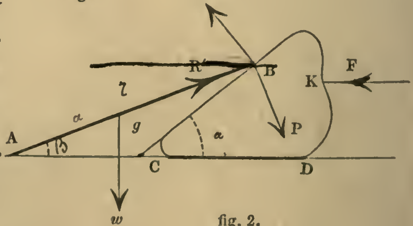
$$\text{or, } w'\sin.\,\theta-w\frac{a}{b}\sin\alpha\,.\,\cos.\,\overline{\alpha+\theta}=0$$

$$\text{whence } \tan.\,\theta=\frac{w\,a\cos.\,\alpha\,.\sin.\,\alpha}{w\,a\sin.^2\alpha+w'b}$$

which gives θ, the elevation of the plane as required.

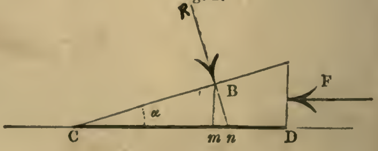
Ex. 13. A heavy beam turns about a hinge at the lower end, with the other end pressing on an inclined plane, a part of the surface of a body which rests on a smooth horizontal plane passing through the hinge; find the horizontal force necessary to keep the body from moving.

fig. 1.



Let BCD be the body resting on the smooth horizontal plane ACD. Let AB be the beam turning about the hinge at A; let g be the center of gravity of the beam, at which its weight (w) acts.

fig. 2.



The body is to be in equilibrium from the pressure of the beam upon it at B, and a horizontal force (F) acting at some point K; and the beam is to be in equilibrium from the reaction (R) of the inclined plane upon it at B, and its own weight acting at g.

The point B is in equilibrium, from the reaction (R) perpendicular to the inclined plane, the reaction (R') of the beam in the direction of its length, and a force (P) acting perpendicularly to AB, arising from the moment of its weight (w) about A.

Taking the moments about A for the equilibrium of the beam, we avoid expressions involving R', and have

$$w \,.\, Ag \cos.\, gAC - R \,.\, AB \sin.\, ABR = 0$$

Let the angle $BAC = \beta$, the angle of the inclined plane with $CD = \alpha$, $Ag = a$, $AB = l$;

$$\text{then } R = w \frac{a \cos.\, \beta}{l \cos.\, (\alpha - \beta)} \qquad (1)$$

To find R in terms of F we must consider the conditions of equilibrium of a right-angled triangular wedge sliding along a smooth plane, as CD, figure 2, from the action of a force (F) acting parallel to CD, a pressure (R) perpendicular to CB at B, and the reaction of the plane CD. If Bn be perpendicular to CB, and Bm to CD, these three forces will be proportional to the sides of the triangle Bmn respectively.

Resolving parallel to CD, we have

$$R \sin.\, BCD - F = 0$$

$$\text{or, in figure 1, } R = \frac{F}{\sin.\, \alpha}$$

Substituting in (1), we have

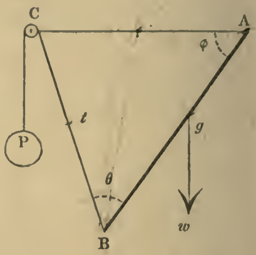
$$F = w \frac{a}{l} \cdot \frac{\sin.\, \alpha \,.\, \cos.\, \beta}{\cos.\, (\alpha - \beta)}$$

which gives the horizontal force required.

Ex. 14. Solve the last example by taking the conditions of equilibrium at the point B; and shew that the whole pressure on the hinge $A = w \left\{ \sin.\, \beta + \frac{a}{l} \cos.\, \beta \,.\, \tan.\, (\alpha - \beta) \right\}$.

Ex. 15. A beam turning about a hinge is supported in equilibrium by the tension in a cord tied to its lower end: the cord passes over a pulley in the same horizontal line with the hinge, and sustains a given weight; find the position of equilibrium of the beam.

In the previous examples we obtained the solution from the equations for equilibrium only, but many statical problems require, for the determination of all the unknown quantities, equations to be formed from *geometrical conditions* also, of which this simple problem is an example.



Let A be the hinge, C the pulley, and $AC = c$.

Let AB be the beam, whose length $= l$, g its center of gravity at which its weight (w) acts, and $Ag = a$.

Let P be the weight hung from the cord, which is equal to the tension (t) in the cord.

Let θ=angle CBA, ϕ=angle CAB; these are both unknown quantities.

Taking moments about A, we have

$$t \,.\, AB \sin.\theta = w \,.\, Ag \cos.\phi$$

$$\text{or, } \sin.\theta = \frac{w}{P} \cdot \frac{a}{l} \cdot \cos.\phi \qquad (1)$$

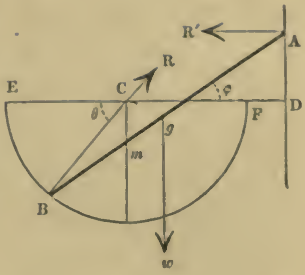
From the *geometrical* data we have

$$\frac{\sin.\theta}{\sin.ACB} = \frac{AC}{AB}$$

$$\text{or, } \sin.\theta = \frac{c}{l} \cdot \sin.(\theta + \phi) \qquad (2)$$

The equations (1) and (2) suffice to determine θ and ϕ.

Ex. 16. A uniform beam rests with its lower end in a smooth hemispherical bowl, and its upper end against a smooth vertical plane; find the position of equilibrium.



Let AB be the beam resting against the vertical plane at A, and upon the bowl at B.

Let C be the center of the bowl; ECF a horizontal diameter which, being produced, meets the vertical plane at D.

Let the radius of the bowl $= r$; $AB =$ length of the beam $= l$; $Ag = \frac{l}{2}$, since the beam is uniform; $w =$ its weight; let also $CD = d$; these must be given.

Let angle $BCE = \theta$, and let $\phi =$ angle of the beam with the horizon; these have to be determined.

The beam is supported by its weight (w) acting at g, the reaction (R') of the plane perpendicular to itself at A, and the reaction (R) in the radius CB.

Resolving vertically, we have

$$R \,.\, \sin.\, \theta - w = 0$$

$$\text{or, } R = \frac{w}{\sin.\, \theta}$$

Taking the moments about A, we have

$$R \,.\, AB \,.\, \sin.\, (\theta - \phi) - w \,.\, Ag \,.\, \cos.\, \phi = 0$$

$$\text{therefore } \frac{\sin.\, (\theta - \phi)}{\sin.\, \theta} - \frac{\cos.\, \phi}{2} = 0 \qquad (1)$$

This equation containing two unknown quantities, we require still a *geometrical relation* between them.

Let Cm be a vertical line meeting AB in m;

$$\frac{\cos.\, \theta}{\cos.\, \phi} = \frac{Bm}{BC} = \frac{AB - Am}{r}$$

$$= \frac{l - d \,.\, \sec.\, \phi}{r}$$

$$\therefore \cos.\, \theta = \frac{l \,.\, \cos.\, \phi - d}{r} \qquad (2)$$

From (1) we have $\cos.\phi - \cot.\theta.\sin.\phi - \frac{1}{2}\cos.\phi = 0$

$$\text{or,}\ \tan.\theta = 2\tan.\phi$$

These two equations suffice to determine ϕ and θ as required.

Ex. 17. A weight (w) hangs from one end of a cord, of which the other end is fastened to a vertical wall: the cord is pushed from the wall by a rod tied to it, which is perpendicular to the wall. Shew that if the cord, where it is fixed to the wall, makes an angle α with it, and R be the pressure of the rod on the wall, then

$$R = w.\tan.\alpha$$

Ex. 18. A heavy beam lies with its upper end against a smooth vertical plane, and its lower end on a smooth horizontal one. Shew that if the beam makes an angle α with horizontal direction, its length being l, and weight w, and the distance of its center of gravity from the lower end being a; then the force required to be applied horizontally at its lower end to maintain the equilibrium being F, we have $F = w\frac{a}{l}\cot.\alpha =$ the pressure against the vertical wall. Shew also that the pressure on the horizontal plane $= w$.

Ex. 19. A body is suspended by a cord of given length from a point in a horizontal plane, and is thrust out of its vertical position by a rod, without weight, acting from another point in the plane; shew that if $t =$ the tension in the cord, $w =$ the weight of the body, $l =$ the length of the cord, $d =$ the distance of the two points, and θ be the angle which the rod makes with the horizon,

$$t = w\frac{l}{d}\cot.\theta$$

Ex. 20. A triangular plate of uniform thickness and density is supported horizontally by a prop at each angle; shew, by drawing perpendiculars on the sides respectively, from the opposite angles and the center of gravity, that the pressure on each prop $= \frac{1}{3}$ the weight of the plate.

CHAPTER VI.

ON THE ELEMENTARY MACHINES, OR MECHANICAL POWERS.

The effects of forces in practical mechanics are continually modified through the agency of instruments which we call machines.

The simplest of these instruments are *Cords* and *Rods*, which, with *hard planes*, may be considered as forming, by their combinations and recombinations, all other machines, however complicated.

Cords are considered in the first instance as without weight, and perfectly flexible. A cord transfers the action of a *pulling force*, applied at one extremity, to any other point in it, unchanged in magnitude, as long as it is in a straight line to that other point, or only passes over smooth obstacles without friction. The force which is thus transmitted along the cord we call the *tension* in the cord.

Rods are considered in the first instance as without weight, and inflexible or rigid. They transmit the action of either *a pulling or a pushing force* in the line joining their extremities unchanged in magnitude. The force which is transmitted along this line we call the *reaction* of the rod.

The machines which are next in simplicity to simple cords and rods are called the *Mechanical Powers*. They comprise the Lever, the Wheel and Axle, Toothed Wheels, the Pulley, the Inclined Plane, the Wedge, and the Screw.

ON THE LEVER.

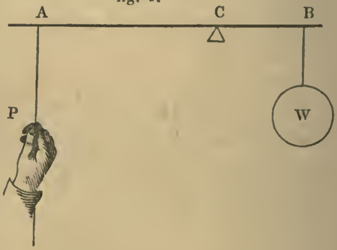
The *simple lever* is a straight rod, having a fixed point somewhere in its length, and supposed without weight. The fixed point about which *the lever* may freely turn is called its *fulcrum*.

The conditions of equilibrium of any *heavy lever* may be reduced to those of a lever without weight, by taking the weight of the lever itself, acting at its center of gravity, with the other forces producing equilibrium.

The *arms* of a lever are the portions of it on each side of the fulcrum. When the arms are not in the same straight line, it is called a *bent lever*.

The mechanical powers being most familiar to us as employed to support or raise heavy bodies or weights, it is usual to call one of the forces the *Power*, and the other the *Weight*.

fig. 1.



Levers have been divided into three kinds, according to the relative positions of the *Power*, the *Weight*, and the *Fulcrum*.

Figure 1 is an example of a lever of the *first kind*, *AB* being the lever, *C* its fulcrum; the power (*P*) and weight (*W*) acting on opposite sides of the fulcrum.

fig. 2.

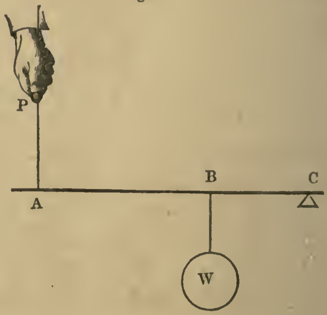


Figure 2 is an example of a lever of the *second kind*, *AC* being the lever, *C* the fulcrum; the power (*P*) and the weight (*W*) acting on the same side of the fulcrum, but *W* nearer to it.

fig. 3.

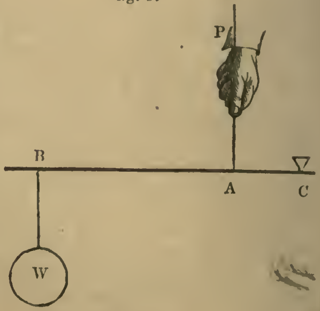


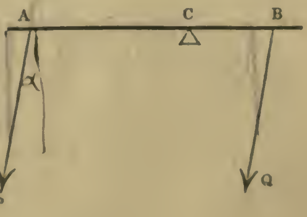
Figure 3 is an example of a lever of the *third kind*, *BC* being the lever, *C* the fulcrum; the power (*P*) and the weight (*W*) acting on the same side of the fulcrum, but the power nearer to it.

A crow-bar, according to the way in which it is used, is a lever of the *first* or *second* kind. *Scissors* and *carpenter's pincers* are examples of *double* levers of the *first* kind. An *oar* is an example of a lever of the *second* kind, the fulcrum being a point in the blade of the oar which rests for an instant stationary in the water. *Nut-crackers* are double levers of the *second* kind.

Tongs, shears, &c., are double levers of the *third* kind. The *bones of the arm* act as levers of the *third* kind.

37. Prop. *To find the condition of equilibrium when two parallel forces act upon a straight lever, and to find the pressure on the fulcrum.*

Since the fulcrum is a fixed point, the only effect of either of the forces tends to turn the lever round this point, and in equilibrium, the tendency to turn it one way round must be balanced by an equal tendency to turn it the other way round; or, the moments of the forces about the fulcrum must be equal and opposite.



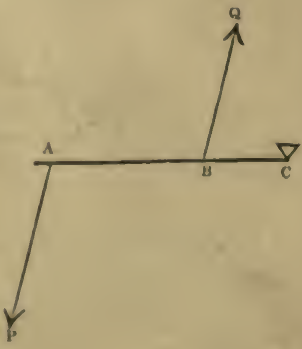
If α be the angle which they make with it, we must have, in both figures,

$$P.AC \sin.\alpha = Q.BC \sin.\alpha$$

$$P.AC = Q.BC$$

$$\text{or, } \frac{P}{Q} = \frac{BC}{AC}$$

which being independent of α, there will be equilibrium in every inclination of the lever to the forces if there be equilibrium in any one; and *the forces are inversely as the distances from the fulcrum at which they act.*



We may solve this proposition by going through all the steps of Article 12, Chap. II.; for, in equilibrium, the resultant of the two parallel forces *must* pass through the fulcrum, and be destroyed by its reaction; therefore the pressure on the fulcrum is always equal to the algebraic sum of the parallel forces; and acts in the direction of the greater force, when they are opposite.

38. Prop. *To find the condition of equilibrium when any two forces in the same plane act upon a straight lever, and the pressure on the fulcrum.*

fig. 1.

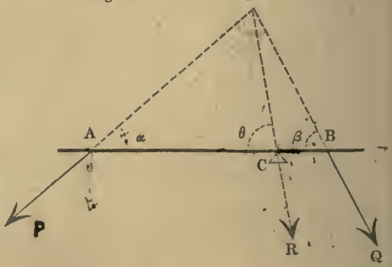
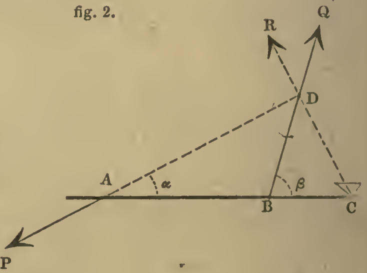


fig. 2.



Let the forces P and Q make the angles α and β respectively with the lever, as in the figures, and let their directions when produced, if necessary, meet in D. Since their moments about C must be equal and opposite when there is equilibrium, we must have

$$P.AC\sin.\alpha=Q.BC\sin.\beta$$

For the pressure on the fulcrum and its direction we must find the magnitude (R) of the resultant of P and Q, and the angle it makes with AB; since in equilibrium it must be destroyed by the reaction of the fulcrum.

By Article 7 we have, in figure 1,

$$R^2=P^2+Q^2+2PQ\cos.ADB$$

$$\text{and } ADB=180-\alpha-\beta$$

$$\therefore R^2=P^2+Q^2-2PQ\cos.(\alpha+\beta)$$

In figure 2, $ADB=\beta-\alpha$, and ADQ is the angle between the forces.

$$\therefore R^2=P^2+Q^2-2PQ\cos.(\beta-\alpha)$$

To find the inclination (θ) of R to the lever, in figure 1. By resolving parallel and perpendicular to the lever, taking R the reaction of the fulcrum opposite to the resultant of the forces, we have

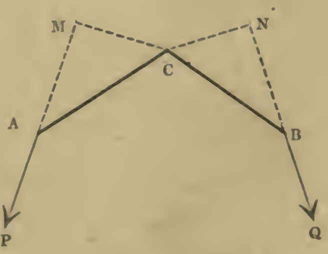
$$P \cos. \alpha - Q \cos. \beta + R \cos. \theta = 0 \qquad (1)$$
$$P \sin. \alpha + Q \sin. \beta - R \sin. \theta = 0 \qquad (2)$$

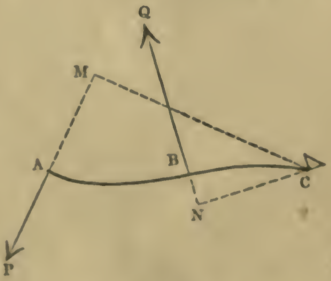
From these equations (1) and (2) we have

$$\tan. \theta = \frac{P \sin. \alpha + Q \sin. \beta}{Q \cos. \beta - P \cos. \alpha}$$

which gives θ. A similar expression may be found for the lower figure.

39. Prop. *To find the condition of equilibrium when two forces in the same plane act in any manner on a lever of any form.*





If P and Q, acting as in the figures at A and B respectively, be in equilibrium about the fulcrum C, and we draw perpendiculars, CM, CN, upon their directions, we have, by the equality of moments about C,

$$P \times CM = Q \times CN$$

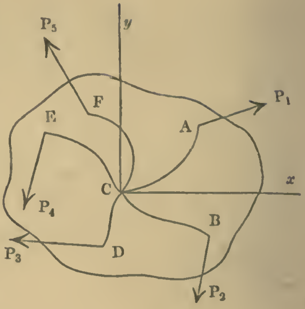
$$\text{or, } \frac{P}{Q} = \frac{CN}{CM}$$

or, in equilibrium the forces are inversely as the perpendiculars upon their directions from the fulcrum.

When the directions of P and Q, with respect to any given straight line through C, are known, the magnitude and direction of the pressure on the fulcrum can be found as in the last proposition.

40. Prop. *To find the condition of equilibrium and the pressure on the fulcrum when any number of forces act in any manner, in one plane, on a lever of any form.*

Let P_1, P_2, P_3, &c. be the forces acting in the plane of the figure at the arms CA, CB, CD, &c. respectively, or at the points A, B, D, &c. in a plane turning about an axis through C perpendicular to it.



Take any two lines perpendicular to each other through C, as Cx, Cy, for the axes of co-ordinates.

Let P be any one of the forces which makes the angle α with Cx, and let x, y be the co-ordinates of its point of application. Then proceeding as in Article 24, we shall have,

$$\text{at } C, \text{ in } Cx, \text{ a force } = \Sigma(P\cos.\alpha)$$

$$\text{at } C, \text{ in } Cy, \quad . \quad . \quad = \Sigma(P\sin.\alpha)$$

and a resultant couple whose moment must $= 0$ when there is equilibrium, or we must have

$$\Sigma(P\cos.\alpha.y - P\sin.\alpha.x) = 0$$

The resultant force at C will be destroyed by the reaction of the fulcrum, and we have the pressure (R) upon the fulcrum from the equation

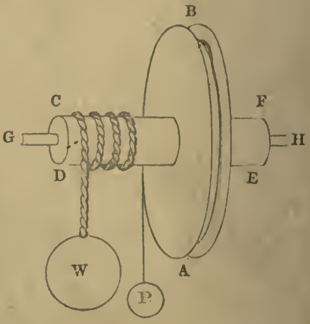
$$R = \sqrt{\{\Sigma(P\cos.\alpha)\}^2 + \{\Sigma(P\sin.\alpha)\}^2}$$

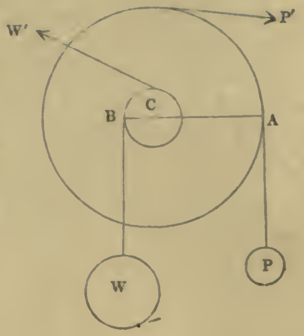
If R makes an angle θ with Cx, we have

$$\tan.\theta = \frac{\Sigma(P\sin.\alpha)}{\Sigma(P\cos.\alpha)}$$

ON THE WHEEL AND AXLE.

41. This machine consists of a wheel, AB, firmly fixed to a cylinder or axle, $CDEF$, with a common axis, GH. The pivots or extremities G and H of the axis generally turn in steps which support the whole; and the forces act by cords which are wrapped round the wheel and the axle.





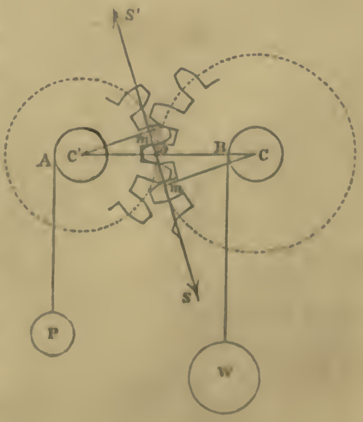
The annexed figure being an end view, we see that by drawing a horizontal line through the center of the axis C, we have the power (P) and the weight (W) acting at A and B, the extremities of a straight lever ACB; and in equilibrium we must have their moments about C equal and opposite; or,

$$P \times AC = W \times BC$$

$$\text{or, } \frac{W}{P} = \frac{AC}{BC}$$

The same relation will exist if the wheel and axle be turned round either way; so that the machine may be called a *perpetual lever*. Also P' and W' being forces equal respectively to P and W, there will be equilibrium at whatever points the cords leave the wheel and axle respectively, since the moments about C remain the same as before.

ON TOOTHED WHEELS.



42. *Toothed* or *Cogged Wheels* are thin cylinders, on the circumference of which are projections called *teeth* or *cogs*, as in the figure. If two such wheels have their cogs set at equal distances they will work together, and if one be set in motion it will communicate motion to the other through the mutual pressure of the cogs which are at any instant in contact.

Let S be this mutual pressure in the figure, which acts in the line $S'm'mS$, a common normal to the cogs at their point of contact; Cm, $C'm'$ being perpendiculars from the centers C, C' of the wheels upon that line.

Let the power (P) act by a weight from a cord wrapped round an axle with center C' and radius $C'A$, and let the weight (W) act similarly from an axle with center C and radius CB.

Taking the moments about C', we have, in equilibrium,

$$P \times C'A = S \times C'm'$$

and about C,

$$W \times CB = S \times Cm$$

Dividing the latter by the former, we have

$$\frac{W}{P} = \frac{C'A}{CB} \times \frac{Cm}{C'm'}$$

Now, if the axles from which P and W act are of equal radii, the effect of the combination will depend on the cog-wheels only, and we have then

$$\frac{W}{P} = \frac{Cm}{C'm'}$$

Let $Sm'mS$ meet the line joining the centers C, C' in o, then the triangles $C'm'o$, Cmo will be similar and

$$\frac{Cm}{C'm'} = \frac{Co}{C'o} = \frac{W}{P}$$

Let the dotted circles be described with radii Co, $C'o$, these are called the *pitch-lines* of the wheels, which roll uniformly upon each other, when the cogs are made of a proper form. In planning wheel-work these pitch-lines are first laid down, and the places of the cogs marked at the same equal distances upon each; their radii are to be so taken that there may be the requisite number of cogs on each circumference; and as they are at equal distances, their number will be in proportion to the circumference.

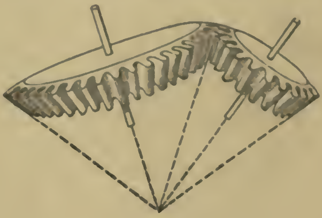
$$\therefore \frac{W}{P} = \frac{Co}{C'o} = \frac{\text{circumference to radius } Co}{\text{circumference to radius } C'o}$$

$$= \frac{\text{number of teeth in wheel of } W}{\text{number of teeth in wheel of } P}$$

The form of the teeth most used in machinery is, that the inner part is formed by radii from the center of the wheel, and the outer part epicycloidal curves; the bottoms of the spaces being rounded to give strength to the teeth. See *Willis's Principles of Mechanism.*

There are several forms of toothed wheels, which are all subject to the above rule. When the teeth project from the flat face of the wheel instead of the edge, they form the *crown wheel.* When the wheel contains very few teeth, it is called a *pinion*, and its teeth or cogs are then called *leaves.* These forms may be seen in most watches.

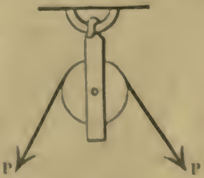
In the preceding instances the axes of the wheels working together were either parallel, as in the first form, or at right angles to each other, as in the *crown wheel* and *pinion.* But wheels are in continual use in which the axes form any given angles, and the cogs are then placed on thin frustums of cones, as in the figure, in place of cylinders; and the wheels are called *bevelled wheels.*



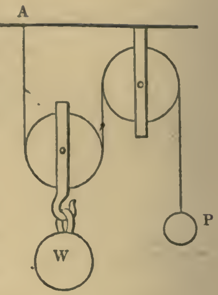
ON THE PULLEY AND SYSTEMS OF PULLIES.

43. The *pulley* is a small wheel, with a groove on its edge to admit a cord which passes over it; it turns round an axis or pivot through its center, which is fastened to the frame-work, called *the block*, in which the pulley moves. It is called a *moveable* or a *fixed* pulley, according as the *block* is *moveable* or *fixed*.

A *fixed pulley* serves merely to change the direction of the force in the cord passing over it; for, neglecting the friction of the pulley, the tension of the cord must be the same in every part.



44. Prop. *To find the relation of the power to the weight in the single moveable pulley with the cords parallel.*



The annexed system, consisting of a moveable pulley from which the weight (W) is suspended, and a fixed pulley over which the cord sustaining the power (P) passes, has the other end fastened at A.

Since the cord is supposed to pass freely over the pullies, the tension will be the same at every point, and the two vertical cords from the moveable pulley sustain W;

$$\text{or, if } t = \text{the tension, } 2t = W = 2P$$

$$\therefore P = \frac{W}{2}$$

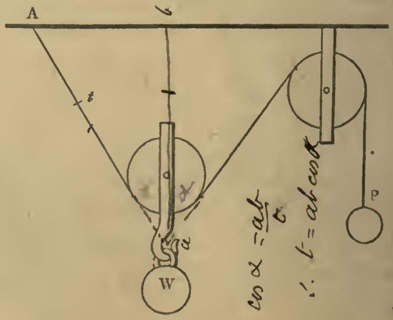
If we take into account the weight of the moveable pulley, we may either add this weight (w) to W, or we may suppose the pulley counterpoised by a weight suspended with P, but not reckoned with it.

Taking $W + w =$ the whole weight sustained by the tension of the two vertical cords, we have

$$2t = W + w$$

$$\text{or, } P = \frac{W + w}{2}$$

45. **Prop.** *To find the relation of the power to the weight in the single moveable pulley with the cords inclined.*



$\cos\alpha = \frac{at}{t}$

$\therefore t = at\cos$

Let $t =$ the tension in cord, which, being fastened at A, passes freely over the moveable pulley, sustaining the weight (W), and the fixed pulley; and is therefore the same at every part of the cord, and equals the power (P).

Let $w =$ the weight of the moveable pulley; $\alpha =$ the angle which the inclined cords make with the vertical direction, and which is the same for both cords, since the resultant of the two equal tensions must be a vertical force sustaining W and the weight of the pulley.

Therefore, resolving in the vertical direction, we have

$$2t \cdot \cos. \alpha = W + w$$

$$\text{or, } P = \frac{W+w}{2 \cos. \alpha}$$

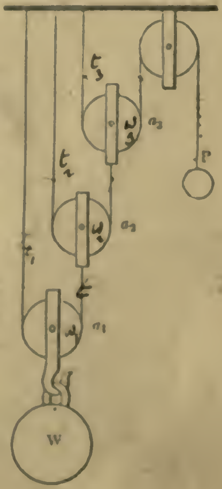
If w be neglected, or counterpoised independently, we have

$$P = \frac{W}{2 \cos. \alpha}$$

We note here, that the two parts of the cord can never be drawn into one straight line, or α become 90°, whilst P and W remain finite, however great P may be, and however small W may be.

Amongst the various ways in which pullies can be combined, there are three, which have been called the first, second, and third systems of pullies.

46. *To find the relation of the power to the weight in a combination of pullies, with the cords parallel, where the power at each pulley acts as weight to the next above it.* This is called the first system of pullies.



The figure represents this system with three moveable pullies, a_1, a_2, and a_3. Let W be the weight, supported at the block of the pulley a_1, and P the power, acting at the last cord after passing over the fixed pulley.

Let w_1, w_2, w_3 be the weights of the pullies a_1, a_2, a_3 respectively. Let t_1 be the tension in the cord passing round the pulley a_1, t_2 that in the cord round a_2, t_3 that in the cord round a_3.

Then, for the equilibrium of a_1, we have

$$2t_1 = W + w_1$$

$$\text{or, } t_1 = \frac{W}{2} + \frac{w_1}{2}$$

For the equilibrium of a_2 we have

$$2t_2 = t_1 + w_2$$

$$\text{or, } t_2 = \frac{t_1}{2} + \frac{w_2}{2}$$

$$= \frac{W}{2^2} + \frac{w_1}{2^2} + \frac{w_2}{2}$$

For the equilibrium of a_3 we have

$$2t_3 = t_2 + w_3$$

$$\text{or, } t_3 = P = \frac{W}{2^3} + \frac{w_1}{2^3} + \frac{w_2}{2^2} + \frac{w_3}{2}$$

which gives the relation of P to W as required.

If the system contains n moveable pullies, we have, similarly

$$P = \frac{W}{2^n} + \frac{w_1}{2^n} + \frac{w_2}{2^{n-1}} + \frac{w_3}{2^{n-2}} + \&\text{c.} \;\ldots + \frac{w^n}{2}$$

If the pullies are all equal, and their weights each equal to w_1, we have

$$P = \frac{W}{2^n} + \frac{w_1}{2^n}(1 + 2 + 2^2 + \&\text{c.} \;\ldots + 2^{n-1})$$

$$= \frac{W}{2^n} + \frac{w_1}{2^n}(2^n - 1)$$

$$= \frac{W}{2^n} + w_1\left(1 - \frac{1}{2^n}\right)$$

In this system a part of the power is expended in sustaining the pullies. If the pullies be counterpoised independently, we have

$$\frac{W}{P} = 2^n.$$

47. Prop. *To find the relation of the power to the weight in a combination of pullies in two blocks where the same cord passes round all the pullies.* This is called the second system of pullies.

fig. 1.

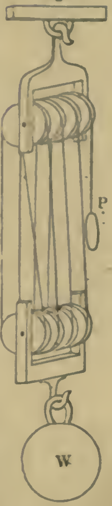
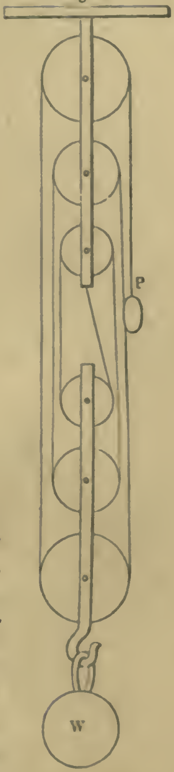


fig. 2.



This combination of pullies is the one in most common use. The pullies are arranged in the blocks in a variety of ways, but figure 1 represents the most usual. *White's Pulley* is a modification of the system, in which all the pullies of each block were formed upon one piece of wood or metal, with their circumferences each proportional to the length of cord which would pass over them when put into use to raise weights, so that all those in each block would have the same angular motion, and they might be then combined into one wheel, by which a great proportion of the friction in the common form would be saved. This plan, which promised so well in a theory constructed upon imperfect data, has been found useless in practice, from that fertile source of error, the omission of essential natural properties, namely, here, the elasticity of all cords, which is continually changing with the wetness or dryness of the atmosphere, and the friction of the cords upon the pullies. Neglecting the friction, we shall have the tension the same at every part of the cord, and counting the number of cords at the lower block, we shall have the number of times the tension which supports the weight (W) and the weight of the block (w).

Therefore, if n = number of cords at the lower block,

$$nt = W + w$$

$$\text{but } t = P$$

$$\therefore P = \frac{W}{n} + \frac{w}{n}$$

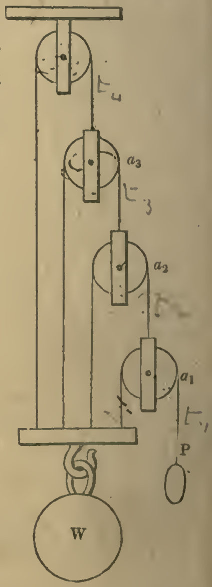
If, as in the figures, the end of the cord be tied to the upper block, the number of pullies in each block will be the same, $=m$, say; and the number of cords at the lower block will be $2m$, and then

$$P=\frac{W}{2m}+\frac{w}{2m}$$

If the weight of the pullies be counterpoised or neglected, we have

$$\frac{W}{P}=n$$

48. Prop. *To find the relation of the power to the weight in a combination of pullies when the cords are parallel, and each attached to the weight.* This is called the third system of pullies.



In this system the uppermost pulley is fixed, and its weight is supported by the beam to which they are attached. Let w_1 be the weight of the pulley a_1; w_2, w_3, those of the pullies a_2, a_3 respectively. Let t_1, t_2, t_3, t_4 be the tensions in the cords respectively, and

$$t_1=P$$
$$t_2=2t_1+w_1=2P+w_1$$
$$t_3=2t_2+w_2=2^2P+2w_1+w_2$$
$$t_4=2t_3+w_3=2^3P+2^2w_1+2w_2+w_3$$

and $W=t_1+t_2+t_3+t_4$

$$=P(1+2+2^2+2^3)+w_1(1+2+2^2)+w_2(1+2)+w_3$$
$$=P(2^4-1)+w_1(2^3-1)+w_2(2^2-1)+w_3(2-1)$$

If the system contains n *moveable* pullies, we shall have, similarly,

$$W=P(2^{n+1}-1)+w_1(2^n-1)+w_2(2^{n-1}-1)+\&c.\ \ldots+w_n(2-1)$$

If the n pullies were of the same weight, w_1, we should have

$$W=P(2^{n+1}-1)+w_1(2^n+2^{n-1}+2^{n-2}+\&c.\ \ldots+2)-nw_1$$
$$=P(2^{n+1}-1)+w_1(2^{n+1}-2-n)$$
$$=(P+w_1)(2^{n+1}-1)-(n+1)w_1$$

In this system the weights of the pullies assist the power. If they are balanced independently, or neglected, we have

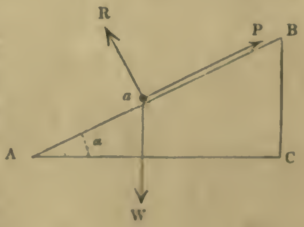
$$\frac{W}{P}=(2^{n+1}-1)$$

ON THE INCLINED PLANE.

When a particle or body is in contact with any hard surface, plane or curved, and the forces press it against the surface, the normal reaction of the surface is one of the forces concurring to produce equilibrium, and must be considered with the other forces.

The inclined plane, as a mechanical power, is supposed perfectly hard and smooth (unless the friction is considered); and having some angle of elevation above the horizontal plane, has a heavy particle or body resting on it in equilibrium, by the action of one or more forces.

49. Prop. *To find the conditions of equilibrium when a body rests on an inclined plane by the action of a force acting up the plane.*



Let the figure represent a section, AB, of the inclined plane by a vertical plane through the body at a. Let AC be a horizontal line, and the angle $BAC=\alpha$.

Let P be the power acting up the plane, R the reaction perpendicular to the plane at a, W the weight of the body acting vertically downwards.

These three forces keep the body in equilibrium; and two out of the three quantities W, P, and α, must be given when the other and R are required. Using the method of Article 23, taking the axis of x in the plane, the axis of y perpendicular to it, with a for origin of co-ordinates, we have the equations of equilibrium

$$\Sigma(X) = 0 \qquad \Sigma(Y) = 0$$

$$\text{or here } P - W \sin.\alpha = 0 \qquad (1)$$

$$R - W \cos.\alpha = 0 \qquad (2)$$

From any point B in the plane draw BC vertical; AB is called the length of the plane, BC its height, and AC its base.

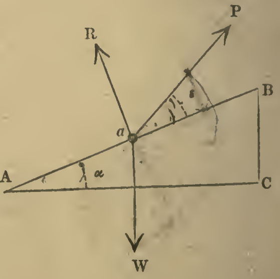
$$\text{From (1), } \frac{W}{P} = \frac{1}{\sin.\alpha} = \frac{AB}{BC}$$

$$= \frac{\text{length of the plane}}{\text{height of the plane}}$$

$$\text{From (2), } \frac{W}{R} = \frac{\text{length of the plane}}{\text{base of the plane}}$$

50. Prop. *To find the conditions of equilibrium when a body is supported on an inclined plane by a force whose direction makes an angle ϵ with the plane.*

P, W, and R being the forces in equilibrium at a, in the figure, where angle $PaB = \epsilon$, angle $BAC = \alpha$. Resolving parallel and perpendicular to the plane, we have, in equilibrium,



$$P \cos.\epsilon - W \sin.\alpha = 0 \qquad (1)$$

$$R + P \sin.\epsilon - W \cos.\alpha = 0 \qquad (2)$$

$$\text{or, } \frac{W}{P} = \frac{\cos.\epsilon}{\sin.\alpha}$$

$$R = W \cos.\alpha - P \sin.\epsilon$$

$$= W \cos.\alpha - W \frac{\sin.\alpha \, . \sin.\epsilon}{\cos.\epsilon}$$

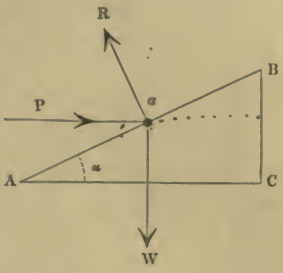
$$= W \frac{\cos.(\alpha + \epsilon)}{\cos.\epsilon}$$

$$\text{or, } \frac{W}{R} = \frac{\cos.\epsilon}{\cos.(\alpha + \epsilon)}$$

If $\alpha + \epsilon = 90$, $P = W$, and $R = 0$.

If $\epsilon > 90^\circ - \alpha$, R is negative, and the body must be on the under side of the plane.

51. PROP. *To find the conditions of equilibrium when a body is supported on an inclined plane by a horizontal pushing force.*



The body being supported on the plane at a by the action of the horizontal force P in Pa; resolving parallel and perpendicular to the plane, we have, in equilibrium,

$$P\cos.\alpha - W\sin.\alpha = 0 \qquad (1)$$

$$R - P\sin.\alpha - W\cos.\alpha = 0 \qquad (2)$$

From (1), $P = W\tan.\alpha$

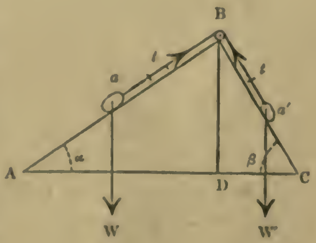
$$\text{or, } \frac{W}{P} = \cot.\alpha = \frac{AC}{BC} = \frac{\text{base of the plane}}{\text{height of the plane}}$$

$$\text{From (2), } R = W\cos.\alpha + P\sin.\alpha$$

$$= \frac{W}{\cos.\alpha}$$

$$\text{or, } \frac{W}{R} = \cos.\alpha = \frac{AC}{AB} = \frac{\text{base of the plane}}{\text{length of the plane}}$$

52. PROP. *To find the relation between the weights of two bodies which rest on two inclined planes having a common summit; the bodies being connected by a cord passing over a pulley at the summit; when they are in equilibrium.*



Let a and a' be the bodies whose weights are W and W'. Let AB, BC be the two planes. Let $\alpha =$ angle BAC, $\beta =$ angle BCA, and BD a perpendicular on AC.

If t be the tension in the cord aBa', we have, for equilibrium on the plane AB,

$$t = W\sin.\alpha$$

and on plane BC,

$$t = W'\sin.\beta$$

$$\therefore W\sin.\alpha = W'\sin.\beta$$

$$\text{or, } W\frac{BD}{AB} = W'\frac{BD}{BC}$$

$$\text{or, } \frac{W}{W'}=\frac{AB}{BC}$$

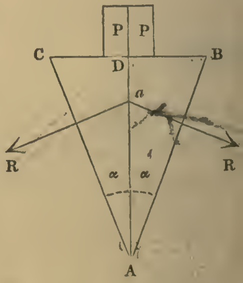
or, the weights are proportional to the lengths of the planes on which they rest respectively.

ON THE WEDGE.

This mechanical power, not less simple than any of the others, has been discussed very differently by different writers, and those who have given the most elaborate solutions have treated it the most erroneously. The student who wishes to understand the mode of action of the wedge should consider attentively the Example 13, page 58.

53. Prop. *To find the conditions of equilibrium on the isosceles wedge.*

By the wedge, we mean a triangular prism whose perpendicular section is an isosceles triangle, as in the figure, where *A* is the section of the *edge* of the prism; *AB*, *AC*, sections of the *sides;* and *CB* of the *back*. The prism is considered hard, and perfectly smooth, if we do not introduce the friction as one of the forces involved.



Draw *AD* bisecting the angle of the wedge, and let $BAD=CAD=\alpha$. Let $2P$ be the power applied at the back of the wedge, which is in equilibrium with the pressures (R) on its two sides, which must be equal, and act through the same point *a* in the power's direction, and also perpendicularly to the sides of the wedge, since they are supposed perfectly smooth.

Resolving in the vertical direction, we have

$$2R\cos. RaA-2P=0$$

$$\text{or, } R=\frac{P}{\sin.\alpha}$$

$$\text{or, } \frac{P}{R} = \sin.\ \alpha = \frac{\frac{1}{2} \text{ back of the wedge}}{\text{side of the wedge}}$$

which gives the force exerted by the wedge perpendicular to its sides.

If there be more than one force acting on each side of the wedge, their resultant in equilibrium must be R, as thus found; and we require such additional data in order to determine the components, as will enable us to solve the triangle of forces when one of the forces (R) is known.

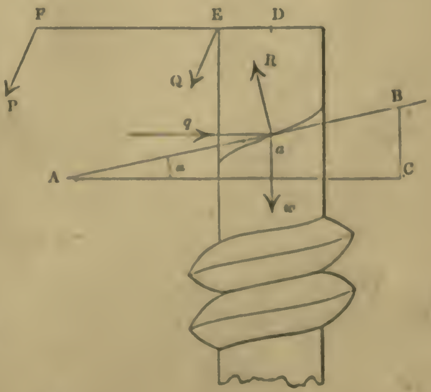
Hatchets, knives, carpenters' chisels, &c. are examples of different forms of the wedge.

ON THE SCREW.

The Screw consists of a *projecting rib* or *thread* passing round a cylinder at the same angle with its axis everywhere. This screw works in a hollow screw to which it fits.

54. Prop. *To find the conditions of equilibrium on the screw.*

If we suppose one revolution of the thread to be unwrapped, it forms an inclined plane, of which the height equals the distance of two threads, and the base the circumference of the cylinder.



Let ABC represent the inclined plane formed by the unwrapping of one revolution of a thread, and BC = the distance of two contiguous threads, AC = circumference of the cylinder.

Let α = angle BAC of the plane, r = radius of the cylinder.

The power P acts perpendicularly at the extremity F of a lever, which turning the screw round, produces a pressure in the direction of the axis of the cylinder of the screw. In equilibrium let this pressure be balanced by a weight W.

Let the arm of the lever of $P=FD=a$. Let Q be the equivalent force to P at the circumference of the cylinder, so that

$$Q=P\frac{FD}{ED}$$

$$=P\frac{a}{r}$$

We may suppose a small part (w) of W to be supported at each point of AB; let q be the corresponding part of Q, and let R be the perpendicular reaction of the thread at a.

Proceeding as in Article 51, we have

$$q=w\tan.\alpha=w\frac{BC}{AC}$$

$$=\frac{w\,.\,\text{distance of two threads}}{\text{circumference of the cylinder}}$$

and the same holds for all other points on the plane; therefore we have

$$Q=W\frac{\text{distance of two threads}}{\text{circumference of the cylinder}}$$

$$=P\cdot\frac{a}{r}$$

$$=P\cdot\frac{2\pi a}{2\pi r}$$

$$\text{or, } P=W\frac{\text{distance of two threads}}{\text{circumference described by } P \text{ in one revolution}}$$

$$\text{or, } \frac{W}{P}=\frac{\text{circumference described by } P}{\text{distance of two threads}}$$

We note here, that the relation of P to W is independent of the radius of the cylinder on which the thread is wrapped.

CHAPTER VII.

ON COMBINATIONS OF THE MECHANICAL POWERS AND BALANCES.

By the *mechanical advantage* of any machine, we mean the number of times the *weight* contains the *power*, or the value of the fraction $\frac{W}{P}$, as used in the preceding propositions.

Recapitulating, we have for the elementary machines the *mechanical advantage :*

$$\text{In the lever} = \frac{\text{the arm of the power}}{\text{the arm of the weight}}$$

$$\text{In the wheel and axle} = \frac{\text{the radius of the wheel}}{\text{the radius of the axle}}$$

$$\text{In toothed wheels} = \frac{\text{the number of teeth in the wheel of } W}{\text{the number of teeth in the wheel of } P}$$

In the single moveable pulley $= 2$

In the first system of pullies $= 2^n$, where $n =$ number of moveable pullies

In the second system of pullies $= n$, where $n =$ number of cords at the lower block

In the third system of pullies $= 2^{n+1} - 1$, where $n =$ number of moveable pullies

$$\text{In the inclined plane} = \frac{\text{the length of the plane}}{\text{the height of the plane}}$$

$$\text{In the wedge} = \frac{\text{the side of the wedge}}{\text{half the back of the wedge}}$$

$$\text{In the screw} = \frac{\text{the circumference described by the power}}{\text{the distance between two contiguous threads}}$$

55. Prop. *To find the mechanical advantage of a combination of any number of elementary machines.*

We may suppose the machines to be connected together by *cords* or *rigid rods*, and the tension or reaction in any cord or rod will be the *weight* to the machine above and the *power* to the machine below. Let the number of machines be n; and let t_1, t_2, t_3, &c. . . . t_{n-1}, be the tensions or reactions in the connecting cords or rods.

Let $P =$ the power for the whole combination.
$W =$ the weight - - - -

The mechanical advantage of the combination $= \frac{W}{P}$

$$= \frac{W}{t_{n-1}} \cdot \frac{t_{n-1}}{t_{n-2}} \cdot \frac{t_{n-2}}{t_{n-3}} \cdot \text{\&c.} \;\ldots\; \frac{t_2}{t_1} \cdot \frac{t_1}{P}$$

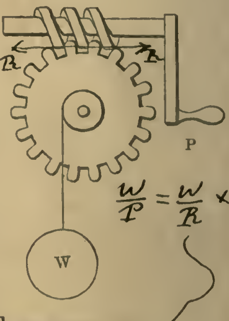
$$= a_n \cdot a_{n-1} \cdot a_{n-2} \cdot \;\ldots\; a_2 \cdot a_1$$

if a_1, a_2, a_3, &c. . . . a_n, be the mechanical advantages of each of the separate machines.

Or, the *mechanical advantage* of a combination of machines equals the product of the mechanical advantages of the separate machines.

56. Prop. *To find the mechanical advantage in the endless screw.*

This machine is a combination of the screw and the wheel and axle. The wheel has projections or teeth on its circumference, set obliquely so as to fit the thread of the screw; the power being applied at the handle of a winch, by which the screw presses against the teeth of the wheel and supports a weight hanging from the axle.



$\frac{W}{P} = \frac{W}{R} \times$

The mechanical advantage of the endless screw =

$$\frac{\text{the circumference described by the power}}{\text{the distance of two threads of the screw}} \times \frac{\text{radius of the wheel}}{\text{radius of the axle}}$$

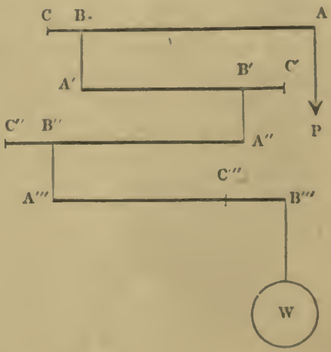
Example. Let the winch be 10 inches, the distance of contiguous threads of the screw $\frac{1}{5}$ inch, the radius of the wheel 10 inches, and that of the axle 1 inch.

$$\text{The mechanical advantage} = \frac{2\pi \times 10}{\cdot 2} \times \frac{10}{1}$$

$$= 3141$$

or, a power of one pound will sustain more than three thousand pounds by such a machine.

57. Prop. *To find the mechanical advantage of a combination of levers.*



Let C, C', C'', C''' be the fulcrums of the levers. Let the power (P) act at A, and the weight (W) at B'''; the last lever being of the first kind, and the other three of the second kind. Let BA', $B'A''$, $B''A'''$ be rigid rods connecting the levers.

The mechanical advantage of the combination =

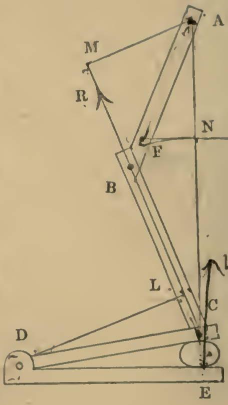
$$\frac{CA \times C'A' \times C''A'' \times C'''A'''}{CB \times C'B' \times C''B'' \times C'''B''}$$

Ex. Let the mechanical advantage of each lever = 10; the mechanical advantage of the combination $= 10^4 = 10000$; or, a power of one pound will sustain a weight of ten thousand pounds by such a combination of levers.

The weighing machine for carts and waggons is a combination of levers; the first being a pair of framework levers which support the platform, on which the cart or waggon is placed when weighed.

58. Prop. *To find the relation of the power to the pressure produced in the combination of levers called* the knee.

A combination of levers like the annexed figure is used with advantage in cases where a very great pressure is required to act through only a very small space, as in coining money, in punching holes through plates of iron, in the printing-press, &c. The lever, AB, turns about a firmly-fixed pivot at A, and is connected by another pivot at B to the rod BC, whose extremity, C, produces the pressure on the obstacle, as E in the figure, being retained in its proper motion by some contrivance producing a similar action to the lever DC in the figure.



Let the power (P) act horizontally at some point F, in the lever AB. Let ANE be a vertical line meeting the direction of P in N, and DE a horizontal plane with the substance subject to pressure at E.

Let $R=$ the reaction of the rod BC in the direction of its length, and AM, DL perpendiculars upon its direction from A and D. Let W be the vertical pressure of the substance at E.

what substance ?

Taking the moments about A and D, in equilibrium we have

$$P.AN=R.AM$$

$$W.DE=R.DL$$

$$\text{or, } W=P\frac{AN}{DE}\cdot\frac{DL}{AM}$$

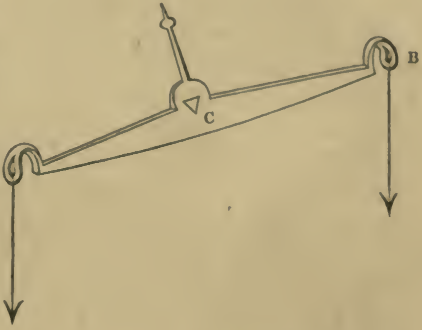
$$\text{or, } \frac{W}{P}=\frac{AN}{AM}\cdot\frac{DL}{DE}$$

When BC comes nearly to the vertical direction, DL comes nearly equal to DE, and AN becomes AF nearly, whilst AM is very small.

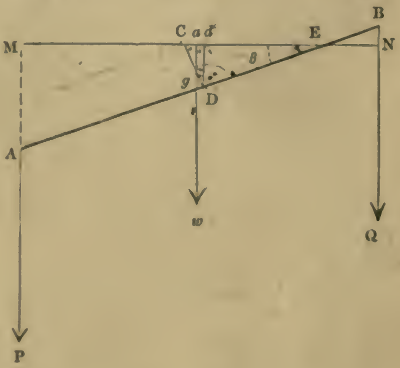
So that we have $\frac{W}{P}=\frac{AF}{AM}$ nearly, which is then very great.

ON THE COMMON BALANCE.

The common balance, as ordinarily constructed, is a bent lever, in which we have to take into consideration the weight of the lever itself.



In the figure, *A* and *B* are the points from which the scale-pans and weights are suspended; *C* is the fulcrum, being the lower edge of a prismatic rod of steel projecting on each side of the beam; when the balance is in use, these edges on each side of the beam, as at *C*, rest on hard surfaces, so that the beam turns freely about *C*, as fulcrum. In the lower figure let *C*, *A*, and *B* be as before; *AB* being the line joining the points of suspension. Draw *CD* a perpendicular on *AB*; when the beam is symmetrical on each side of *C*, its center of gravity will be at some point as *g* in *CD*.



The requisites of a good balance are:

1*st*. That the beam rests in a horizontal position when loaded with equal weights. at each end. [illegible]

2*d*. That the balance possesses great sensibility.

3*d*. That it possesses great stability.

For the first condition, it is necessary that the arms are of equal lengths, and that the beam is symmetrical on each side of

C, with the points of suspension and the center of gravity below that point. When this is the case, we shall have the perpendicular CD bisecting AB in D.

59. Prop. *To investigate the conditions that a balance may possess great sensibility and great stability.*

A balance possesses great *sensibility*, when, for a *small* difference between the weights P and Q with which it is loaded, the line AB is considerably deflected from the horizontal position in which it rests when loaded with equal weights. Let $M\,C\,E\,N$ be a horizontal line through C, meeting AB in E, and let the angle $MEA=\theta$. The sensibility depends on the magnitude of θ compared with the difference $P-Q$ of the weights. Let w equal the weight of the beam. Let the horizontal line through C meet the vertical lines through A and B in M and N respectively; and those through g and D in a and d respectively. Then MN is bisected in d.

Taking the moments about C, we have in equilibrium

$$P.CM-Q.CN-w.Ca=0$$

$$\text{or, } P(Md-Cd)-Q(Nd+Cd)-w.Ca=0$$

$$\text{or, } (P-Q)AD\cos.\theta-(P+Q)\,CD\sin.\theta-w.Cg\sin.\theta=0$$

Let the length AD or BD of the arms $=a$, $CD=d$, and $Cg=h$.

$$\text{then, } (P-Q)\,a-\{(P+Q)\,d+w.h\}\tan.\theta=0$$

$$\therefore \tan.\theta=\frac{(P-Q).a}{(P+Q)d+w.h}$$

We see that θ, and therefore the sensibility, will be increased for given values of P and Q by increasing a, and by diminishing w, d, and h; or, by increasing the lengths of the arms, by diminishing the weight of the beam, and by diminishing the distances of the fulcrum from the center of gravity of the beam, and from the line joining the points of suspension of the scale-pans.

A balance possesses great *stability*, when, being loaded with equal weights, on being disturbed it returns *quickly* towards its position of equilibrium. The *stability* is therefore greater as the moment bringing the beam towards its horizontal position is greater.

Or, since $P = Q$,

$$\text{as } P.CN - P.CM + w.Ca \text{ is greater;}$$

$$\text{or, as } P(\overline{Nd + Cd} - \overline{Md - Cd}) + w.Ca$$

$$= (2P.d + w.h)\sin.\theta \text{ is greater.}$$

This is greater for given values of P and θ, as d, w, and h are increased.

We see that the *sensibility* of the balance is diminished as we increase the *stability*, but that we may increase the *sensibility* without injuring the *stability* by increasing the length of the arms.

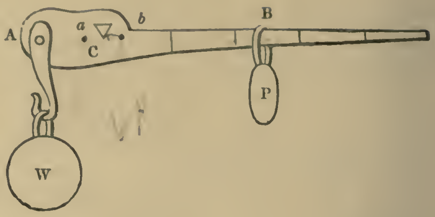
We see also that balances must be adapted to the uses they are to be applied to. The *fine, delicate* balance of the chemical laboratory must possess great sensibility, but we must not expect great stability. For weighing *coarse wares*, it is of more consequence that the balance possesses great stability than that it shews very small differences of weight, where the material weighed is not of great value.

ON THE STEELYARD BALANCE.

The steelyard has a longer and a shorter arm, as in the figure; the substance to be weighed being hung from the point A in the shorter arm, its weight is found from the distance to which the constant weight P must be moved on the longer arm in order to balance it.

60. Prop. *To shew that the divisions on the longer arm of the steelyard, which correspond to equal additions of weight in the body weighed, must be in a succession of equal distances.*

Let the projecting knife-edge at C be the fulcrum, and A



the point from which the weight W is suspended. Let P be the moveable weight. When no weight is suspended from A, the longer arm will preponderate; let a be the point from which P must be suspended to produce equilibrium. Take $Cb = Ca$, then P at b would balance P at a if the lever was without weight, consequently we may consider it without weight if we suppose a weight equal to P to be hung from b. Let the steelyard be in equilibrium from the weight of W at A, of P at B and the effect of the weight of the lever.

From the equality of the moments about C we have

$$W.CA - P.CB - P.Cb = 0$$

$$\text{or, } \frac{W}{P} = \frac{CB + Cb}{AC}$$

$$= \frac{aB}{AC}.$$

$$\text{If } W = P \text{ then } aB = AC$$

$$\text{-}\ W = 2P\ \text{-}\ aB = 2AC$$

$$\text{-}\ W = 3P\ \text{-}\ aB = 3AC$$

$$\text{-}\ W = n.P\ \text{-}\ aB = n.AC$$

Or, for every additional weight P by which W is increased, the moveable weight will have to be moved a distance equal to AC further along the arm to balance it; and for equimultiples of P the divisions will be in a succession of distances equal to AC counted from a.

This same rule holds for increments of W corresponding to any fractional part of P. Let it be required to graduate the steelyard for a succession of weights increasing by $\frac{P}{m}$.

$$\text{From the expression } \frac{W}{P} = \frac{aB}{AC}$$

$$\text{we have } \frac{W}{\frac{P}{m}} = \frac{aB}{\frac{AC}{m}}$$

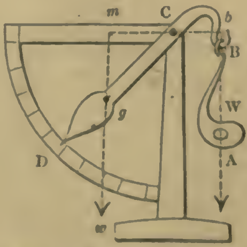
Let $W = N.\frac{P}{m}$, where N is any integer.

$$N = \frac{aB}{\frac{AC}{m}} \qquad \text{or, } aB = N.\frac{AC}{m}$$

Giving N a succession of integral values, we shall have for aB a succession of distances counted from a, increasing by $\frac{AC}{m}$.

It is scarcely necessary to remark, that the method supposed to be followed above would never be the practical method of graduating the steelyard; but the discussion is of use, to shew that the divisions must be at equal distances.

ON THE BENT LEVER BALANCE.



61. This balance is similar to the figure, where BCD is the bent lever, turning about a pivot at C. A scale (A) hangs from B; and at D an index points to some division on the graduated arc.

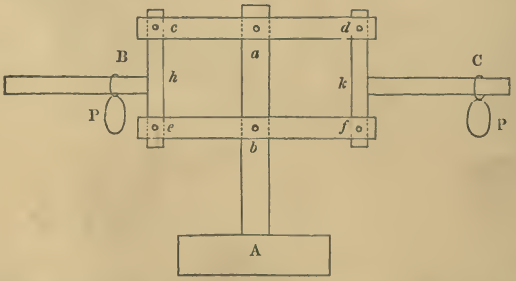
Let g be the center of gravity of the beam at which its weight, w, acts. The weight of the scale (A) and the weight (W) of a body placed in it will act vertically through B. If we draw a horizontal line through C, meeting the vertical lines through g and B respectively in m and b, we must have, in equilibrium,

$$w \,.\, Cm - (A + W) Cb = 0$$

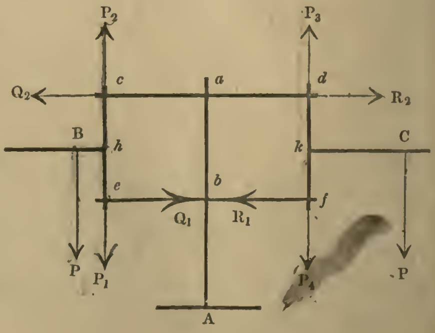
Now, as greater weights are put into the scale A, the point B comes more nearly to the vertical line through C, from the bent form of the beam; and the distance Cm increases, so that the arc may be graduated from the positions of the index at D for a succession of weights put into A. When the arc has been thus graduated experimentally, the weight of a body placed in the scale is told very *quickly* by the division on the arc to which the index rises. When it is required to sort a great number of bodies into classes of different weights, but where *extreme* nicety is not essential, this balance is the most convenient for the purpose.

ON ROBERVAL'S BALANCE.

62. This balance is of greater interest from its paradoxical appearance than from its use as a machine for weighing bodies. It consists of an upright stem upon a heavy base, *A*, with equal crossbeams turning about pivots at *a* and *b*. These crossbeams are connected by pivots at *c*, *d*, *e*, *f*, with other equal pieces in the form of the letter T; the weights are hung from the horizontal arms of the latter pieces; and the peculiar property of the balance is, that *equal* weights balance at *all distances* from the upright stem: thus, two equal weights (*P*), as in the figure, balance; although one may hang at *B* much nearer the upright stem than *C*, the point from which the other hangs.



This property is easily proved from the theory of couples. Let the letters in the annexed figure indicate the same parts as in the former.



Let equal and opposite forces P_1 and P_2, as in the figure, act in *ec*, each equal to P; and similarly let P_3 and P_4 act in df. These forces, P_1, P_2, P_3, P_4, will not affect the equilibrium; and P at B is equivalent to P_1 at e and the couple P, Bh, P_2. Similarly, P at C is equivalent to P_4 at f, and the couple P, Ck, P_3. The forces P_1 at e and P_4 at f will evidently balance.

The couple P, Bh, P_2 is equivalent to a couple Q_1, ec, Q_2 in its own plane, of equal moment, in which we may take the forces Q_1 and Q_2 acting in the directions of the crossbeams eb, ac, which always remain parallel to each other as they turn on the pivots. These forces Q_1 and Q_2 are destroyed by the resistance of the pivots b and a; and similarly, if R_1, fd, R_2 were the corresponding couple on the other side, R_1 and R_2 would be destroyed by the resistance of b and a. These couples therefore would not affect the equilibrium, and the original forces P at B and P at C must be in equilibrium.

If the beams cad, ebf be moved round the pivots into any oblique position, the same reasoning holds good, and the equilibrium still subsists.

It is easy to see that unequal weights, as P and Q, could not balance when hanging from any points.

CHAPTER VIII.

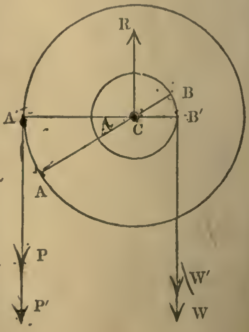
APPLICATION OF THE PRINCIPLE OF VIRTUAL VELOCITIES TO THE MECHANICAL POWERS.

We saw in Articles 26 and 27 that the principle of virtual velocities holds good for all cases of the equilibrium of a free body under the action of any number of external forces.

We may consider the effective parts of the mechanical powers as free bodies if the reactions of the parts which support them be taken with the other external forces; and the internal reactions and tensions do not enter the fundamental expression $\Sigma(P.v) = 0$. We shall also generally have the virtual velocities of the reactions of the supporting parts equal to zero for the possible displacements of the system.

In some of the mechanical powers we have the principle applying to all possible displacements whether great or small, since they are always in the direction of the forces; as in the wheel and axle, toothed wheels, the pullies with parallel cords, the inclined plane, the wedge, and the screw. In the lever and the pullies with inclined cords we must take the displacement indefinitely small.

63. Prop. *To shew that the principle of virtual velocities holds good for the wheel and axle in equilibrium.*



The forces which act on the wheel and axle are the power P, the weight W, and the reaction R of each of the steps which support each end, C, of the pivot about which it turns. When the wheel and axle receive a displacement turning about C, the virtual velocity of R equals 0.

Let A and B be the points at which the cords left the wheel and the axle respectively before displacement; A', B', afterwards. Then W ascends through the space $WW' =$ arc BB', and P descends through $PP' =$ arc AA'. WW' is a negative virtual velocity if PP' be positive.

By the principle of virtual velocities,

$$P \times PP' - W \times WW' = 0$$

$$\text{or, } P \times \text{arc } AA' - W \times \text{arc } BB' = 0$$

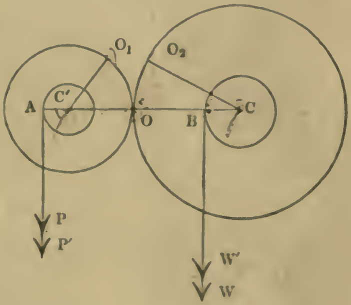
$$\text{or, } P \times AC \times \text{angle } A'CA - W \times BC \times \text{angle } BCB' = 0$$

$$P \times AC - W \times BC = 0$$

$$\text{or, } \frac{W}{P} = \frac{AC}{BC}$$

the condition of equilibrium as found in Article 41.

64. Prop. *To shew that the principle of virtual velocities holds good in a pair of toothed wheels.*



Let the circles in the figures represent the pitch-lines of the wheels which roll on each other without slipping, and let O_1, O_2 be the points which were in contact in the line CC' before disturbance. Then the other letters being as in Article 42, we have

$$\text{arc } OO_1 = \text{arc } OO_2$$

$$P\text{'s displacement} = PP'$$

$$= AC' \times \text{angle } O_1C'O$$

$$= AC' \times \frac{\text{arc } O_1O}{C'O}$$

$$W\text{'s displacement} = WW'$$

$$= CB \times \frac{\text{arc } O_2O}{CO}$$

By the principle of virtual velocities,

$$P \times PP' - W \times WW' = 0$$

$$\text{or, } P\times AC'\times\frac{\text{arc } O_1O}{C'O} - W\times CB\times\frac{\text{arc } O_2O}{CO}=0$$

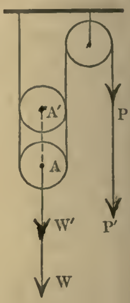
$$P\frac{AC'}{C'O} - W\frac{CB}{CO}=0$$

$$\text{or, } \frac{W}{P}=\frac{AC'}{BC}\cdot\frac{CO}{C'O};\ \text{when } AC'=BC,$$

$$=\frac{CO}{C'O}$$

as found for the condition of equilibrium in Article 42.

65. Prop. *To shew that the principle of virtual velocities holds good in the single moveable pulley with the cords parallel.*



In the figure, if the pulley A be raised to A', we shall have $AA'=WW'=\frac{1}{2}PP'$, since each of the cords passing round the pulley A must be shortened by a length $=WW'$. And WW' is a negative virtual velocity.

$$\therefore\ P\times PP'-W\times WW'=0$$

$$\text{gives } P\times PP'-W\times\tfrac{1}{2}PP'=0$$

$$\text{or, } \frac{W}{P}=2$$

the condition of equilibrium as found in Article 44.

66. Prop. *To shew that the principle of virtual velocities holds good in the first system of pullies.*

Referring to the figure in Article 46, we see that if P descended through a space PP',

the pulley a_n would be raised a space $\frac{1}{2}PP'$

the pulley a_{n-1} - - - $\frac{PP'}{2^2}$

&c. &c.

the pulley a_3 - - - - $\frac{PP'}{2^{n-2}}$

the pulley a_2 - - - - $\frac{PP'}{2^{n-1}}$

the weight W or the pulley a_1 - - - $\frac{PP'}{2^n}$

and the equation of virtual velocities

$$P \times PP' - W \times WW' = 0 \text{ becomes}$$

$$P \times PP' - W \times \frac{PP'}{2^n} = 0$$

$$\text{or, } \frac{W}{P} = 2^n$$

the condition of equilibrium when the weights of the pullies are counterbalanced or neglected.

67. Prop. *To shew that the principle of virtual velocities holds good in the second system of pullies.*

Referring to the figure in Article 47, we see that if the weight be raised through a space WW', each of the n cords at the lower block must be shortened the same quantity, or P must descend through a space $n \times WW'$. The equation of virtual velocities is

$$P \times PP' - W \times WW' = 0$$

$$\text{which becomes } P \times n \,.\, WW' - W \times WW' = 0$$

$$\text{or, } \frac{W}{P} = n$$

the condition of equilibrium as in Article 47.

68. Prop. *To shew that the principle of virtual velocities holds good in the third system of pullies.*

Referring to the figure of Article 48, we see that if W be raised a space $= WW'$, each cord will be shortened in consequence a space equal to it. The highest moveable pulley, a_n, will descend through the same space, WW'. The next pulley, a_{n-1}, will descend through a space $2 \,.\, WW'$, in consequence of the descent of a_n, and WW' in consequence of the elevation of W, or will descend on the whole $(2+1)\,WW'$.

Similarly, the pulley a_{n-2} will descend through

$$\{2(2+1)+1\}\,WW' = (2^2+2+1)\,WW'.$$

Proceeding in the same way, we find that pulley a_3, or, $a_{n-\overline{n-3}}$, will descend through the space

$$(2^{n-3} + 2^{n-4} + \&\text{c.} \;.\;.\;. + 2 + 1)\,WW'$$

and a_2 through the space $(2^{n-2}+2^{n-3}+\&c. \; . \; . \; . +2+1)\, WW'$
a_1 - . $(2^{n-1}+2^{n-2}+\&c. \; . \; . \; . +2+1)\, WW'$

P descends through twice the last found space in consequence of the descent of the pulley a_1, and through the space WW', in consequence of the elevation of the weight;

$$\text{or, } PP' = WW' \,\{2(2^{n-1}+2^{n-2}+\&c. \; . \; . \; . +2+1)+1\}$$
$$= WW'\,(2^{n+1}-1)$$

The equation of virtual velocities is

$$P \times PP' - W \times WW' = 0$$

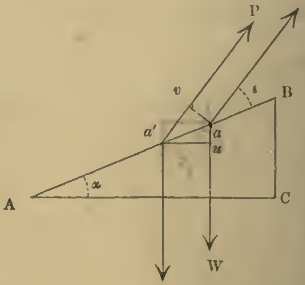
$$\text{which becomes } P \times WW'\,(2^{n+1}-1) - W \times WW' = 0$$

$$\text{or, } \frac{W}{P} = 2^{n+1}-1$$

the condition of equilibrium when there are n moveable pullies, as in Article 48.

69. Prop. *To shew that the equation of virtual velocities holds in the inclined plane.*

Taking the most general case, when the force (P) makes any angle ϵ with the plane. Let $\alpha = \angle BAC$; a the first position of the body whose weight is W; a' the position of it after a disturbance.



Drawing the perpendiculars av, $a'u$, we have $-a'v$, the virtual velocity of $P = -aa' \cos. \epsilon$, and au, the virtual velocity of $W = aa' \sin. \alpha$.

By the equation of virtual velocities,

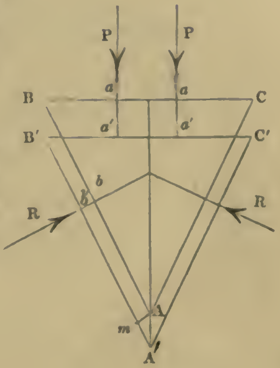
$$P \times a'v - W \times au = 0$$

$$\text{or, } P \times aa' \cos. \epsilon - W \times aa' \sin. \alpha = 0$$

$$\text{or, } \frac{W}{P} = \frac{\cos. \epsilon}{\sin. \alpha}$$

as found for the condition of equilibrium in Article 50.

70. Prop. *To shew that the principle of virtual velocities holds for the wedge.*



Let $2P$ be the whole power, R and R the pressures perpendicular to the smooth sides of the wedge ABC, which produce equilibrium.

Let the wedge be displaced to the position $A'B'C'$. The displacement of the point of application of P is aa' in the figure, $=AA'$; that of b, the point of application of R, is $bb'=Am$, a perpendicular from A on $A'B'$; and

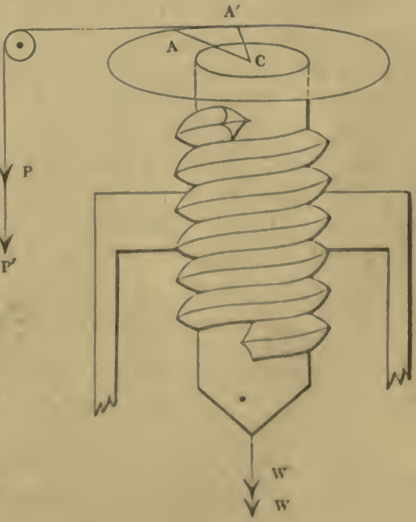
$$Am = AA' \text{ sin. } \frac{BAC}{2}$$

The equation of virtual velocities is

$$P \times aa' - R \times bb' = 0$$

$$\text{or, } P \times AA' - R \times AA' \text{ sin. } \frac{BAC}{2} = 0$$

$$\therefore R = \frac{P}{\text{sin. } \frac{BAC}{2}}$$



as found in Article 53, for the condition of equilibrium of the wedge.

71. Prop. *To shew that the principle of virtual velocities holds for the screw.*

Suppose the power P to act by means of a cord passing over a pulley upon a wheel, as perpetual lever, fixed to the cylinder of the screw.

Let A be the point where the cord left the wheel when the power was at P and the weight at W; and let A', P', W' be their positions after a disturbance,

$$PP' = \text{arc } AA' = AC \times \text{angle } ACA'$$

$$WW' = \text{distance of two threads} \times \frac{\text{angle } ACA'}{2\pi}$$

By the equation of virtual velocities,

$$P \times PP' - W \times WW' = 0$$

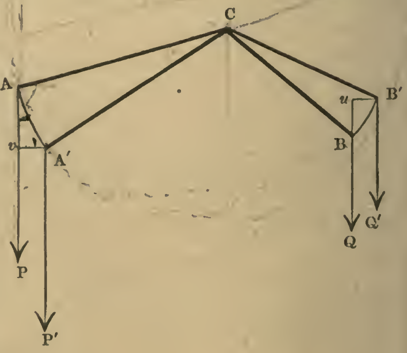
or, $P \times AC \times \text{angle } ACA' - W \times \text{distance of two threads} \times$

$$\frac{\text{angle } ACA'}{2\pi} = 0$$

$$\text{or, } \frac{W}{P} = \frac{2\pi AC}{\text{distance of two threads}}$$

the condition of equilibrium as found in Article 54.

72. PROP. *To shew that the principle of virtual velocities holds in a lever of any form.*



Let ACB be the lever before displacement, $A'CB'$ its position after.

From A' draw $A'v$ perpendicular to AP, and from B', $B'u$ perpendicular to BQ produced. Av is the virtual velocity of P, Bu that of Q, and negative. Now, when the displacement is indefinitely small, the circular arcs AA', BB' become straight lines, and

$$Av = AA' \times \cos. A'Av = AC \times \text{angle } ACA' \times \cos.(PAC - 90^\circ)$$

$$= AC \,.\, \sin. PAC \,.\, \text{angle } ACA'$$

$$Bu = BC \,.\, \sin. QBC \,.\, \text{angle } BCB'$$

$$\text{and angle } ACA' = \text{angle } BCB'$$

The equation of virtual velocities is

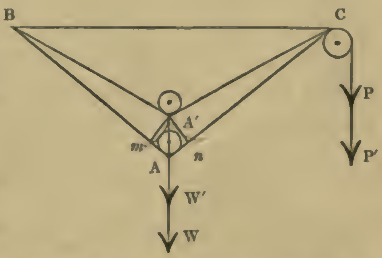
$$P \times Av - Q \times Bu = 0$$

$$\text{or, } P \, . \, AC \, . \sin. PAC - Q \, . \, BC \, . \sin. QBC = 0$$

$$\text{or, } \frac{P}{Q} = \frac{BC \, . \sin. QBC}{AC \, . \sin. PAC}$$

as found in Article 39 for the condition of equilibrium.

73. Prop. *To shew that the principle of virtual velocities holds in the single moveable pulley with the cords inclined.*



Let A be the point where the cords produced would meet at the first position of the pulley, when P and W are the positions of the power and the weight.

Let P be displaced to P' when the weight is raised to W', or the point of meeting of the cords to A'. Draw the circular arcs, $A'm$, $A'n$, with centers B and C. When the displacement is indefinitely small, the arcs $A'm$, $A'n$ become straight lines, and

$$Am = AA' \cos. BAA' = An$$

$$PP' = Am + An = 2AA' \cos. \frac{BAC}{2}$$

$$WW' = AA'$$

The equation of virtual velocities is

$$P \times PP' - W \times WW' = 0$$

$$\text{becomes } P \times 2AA' \cos. \frac{BAC}{2} - W \times AA' = 0$$

$$\text{or, } \frac{W}{P} = 2 \cos. \frac{BAC}{2}$$

as found for the condition of equilibrium in Article 45.

In the preceding propositions, the expression

$$P \times PP' = W \times WW'$$

$$\text{or, } \frac{WW'}{PP'} = \frac{P}{W}$$

explains the principle that, “ in using any machine, what we gain in power we lose in time." For, in order that W may be raised through any given space, we must have the space moved through by P increased in the same ratio that the magnitude of P is diminished.

EXAMPLES ON CHAPTERS VI. VII. AND VIII.

Ex. 1. A lever 30 feet long balances itself upon a prop $\frac{1}{3}$ of its length from the thicker end; but when a weight of 10 pounds is suspended at the other end, the prop must be moved 2 feet towards it, to maintain the equilibrium: shew that the weight of the beam is 90 pounds. $W \times 2 = 18 \times 10$

Ex. 2. The forces P and Q act at arms a and b respectively of a straight lever which rests upon a fixed point to which it is not attached. When P and Q make angles α and β with the lever, shew that the conditions of equilibrium are

$$P \cos. \alpha + Q \cos. \beta = 0$$

$$P a \sin. \alpha - Q b \sin. \beta = 0$$

Ex. 3. The beam of a false balance being of uniform density and thickness, it is required to shew that the lengths of the arms (a and b) are respectively proportional to the differences between the true (W) and apparent weights (P and Q). Or to shew that the weight of the beam being considered, we have

$$\frac{a}{b} = \frac{P - W}{W - Q}$$

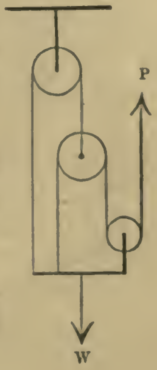
Ex. 4. Two given weights hanging vertically from two given points in the rim of a wheel, find the position in which the greatest weight will be sustained on the axle.

Let P and Q be the weights, α the angle contained between the radii drawn to the points of suspension, which are given. Let θ be the angle which the lower radius (that of P) makes with the vertical direction. It is shewn by means of a subsidiary angle, and without the differential calculus, that

$$\tan. \theta = \frac{P + Q \cos. \alpha}{Q \sin. \alpha}$$

Ex. 5. If l be the mechanical advantage of a lever, and s the mechanical advantage of a screw, shew that the mechanical advantage of the *common vice* is $l \,.\, s$.

Ex. 6. Apply the principle of virtual velocities to find the relation of the power to the weight in the *endless screw*, as found in Article 56.

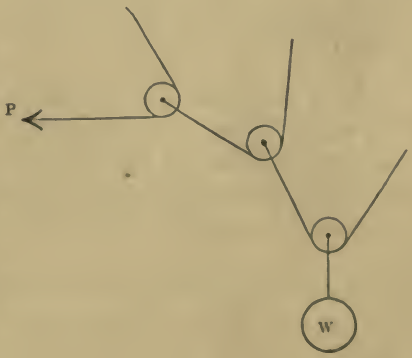


Ex. 7. Shew that, in the annexed system of pullies, $W = 5P$. Apply the principle of virtual velocities to find the same result.

Ex. 8. In the annexed system of three moveable pullies, with the cords at each pulley inclined 60° to each other, shew that

$$W = P(3)^{\frac{3}{2}}$$

Obtain the same result by means of the principle of virtual velocities.

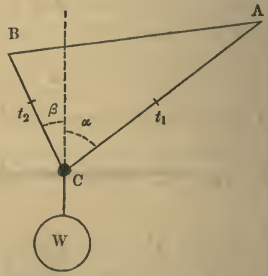


Ex. 9. Shew that in every combination of the mechanical powers, the relation of the power to the weight may be found by the principle of virtual velocities.

CHAPTER IX.

ON THE EQUILIBRIUM OF A SYSTEM OF WEIGHTS SUSTAINED BY CORDS OR BEAMS.

74. *If a weight W be hung from a knot C at which the cords AC, BC (supposed without weight) meet, and the lengths of the cords and the position of A and B are given; to find the tensions in the cords.*



The geometrical data give the angles which AC and BC make with the vertical direction; let them be α and β respectively.

If t_1 and t_2 be the tensions in the two cords AC and BC respectively, we may find their values by the properties of the parallelogram of forces or otherwise analytically as in Article 23, as follows:

Resolving horizontally and vertically,

$$t_1 \sin.\alpha - t_2 \sin.\beta = 0 \qquad (1)$$

$$t_1 \cos.\alpha + t_2 \cos.\beta - W = 0 \qquad (2)$$

Multiply (1) by cos. β, (2) by sin. β, and add, we find,

$$t_1(\sin.\alpha.\cos.\beta + \cos.\alpha.\sin.\beta) - W.\sin.\beta = 0$$

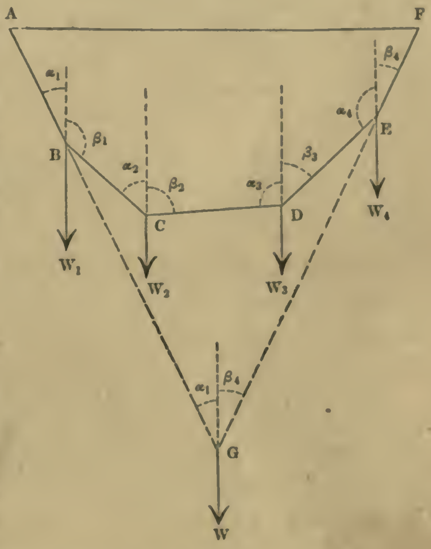
$$\text{or,}\quad t_1 = \frac{W.\sin.\beta}{\sin.(\alpha+\beta)}$$

Similarly,

$$t_2 = \frac{W\sin.\alpha}{\sin.(\alpha+\beta)}$$

We should find these results at once from Article 6, but the equation (1) shews us a property in addition, which we shall find to hold for systems of any number of cords and beams, namely, that the resolved parts of the tensions *horizontally* are the same in each.

75. Prop. *To investigate the form and properties of the* funicular polygon, *or a system of cords connected by knots from which given weights are hung.*



Let *ABCDEF* be the polygon formed by cords connected by knots at the points *B*, *C*, *D*, *E*. Let *AF* be a horizontal distance which is given, as well as l_1, l_2, l_3, &c. the lengths of the cords *AB*, *BC*, *CD*, &c., which we suppose without weight.

Let W_1, W_2, W_3, &c. be the weights suspended from *B*, *C*, *D*, &c. respectively.

The above being given, we have to determine the tensions in the cords, and the angles they make with the vertical direction.

Let t_1=tension in *AB*, t_2=tension in *BC*, t_3=tension in *CD*, &c.

Let α_1, β_1 be angles of the cords with the vertical direction at *B*.

α_2, β_2 - - - - *C*.

α_3, β_3 - - - - *D*.

&c. - - - - &c.

If n be the number of cords, we have n tensions, and $2(n-1)$ angles to determine, from the following equations:

$$AF - l_1 \sin.\alpha_1 - l_2 \sin.\alpha_2 - \&c. \;.\;.\;. - l_{n-1} \sin.\alpha_{n-1} - l_n \sin.\beta_{n-1} = 0 \qquad (a)$$

$$l_1 \cos.\alpha_1 + l_2 \cos.\alpha_2 + l_3 \cos.\alpha_3 + \&c. \;.\;.\;.\;. + l_{n-1} \cos.\alpha_{n-1} + l_n \cos.(\pi - \beta_{n-1}) = 0 \qquad (b)$$

Resolving horizontally and vertically at each knot, we have

at B, $t_1 \sin.\alpha_1 - t_2 \sin.\beta_1 = 0 \qquad t_1 \cos.\alpha_1 + t_2 \cos.\beta_1 - W_1 = 0$ (1)

at C, $t_2 \sin.\alpha_2 - t_3 \sin.\beta_2 = 0 \qquad t_2 \cos.\alpha_2 + t_3 \cos.\beta_2 - W_2 = 0$ (2)

&c. &c.

$t_{n-1} \sin.\alpha_{n-1} - t_n \sin.\beta_{n-1} = 0 \qquad t_{n-1} \cos.\alpha_{n-1} + t_n \cos.\beta_{n-1} - W_{n-1} = 0$ $(n-1)$

We have also $\beta_1 + \alpha_2 = \pi$, $\beta_2 + \alpha_3 = \pi$, &c. . . $\beta_{n-2} + \alpha_{n-1} = \pi$, which are $n-2$ equations.

We have, consequently, the equations (a) and (b), $2(n-1)$ and $n-2$; or, in all, $3n-2$ equations, to find the n tensions and $2n-2$ angles as required.

Since $\beta_1 + \alpha_2 = \pi$, we have $\sin.\beta_1 = \sin.\alpha_2$, and so onwards, $\sin.\beta_2 = \sin.\alpha_3$, &c. . . . $\sin.\beta_{n-2} = \sin.\alpha_{n-1}$; therefore we see that the first of each pair of the equations (1), (2), . . . $(n-1)$ shews the horizontal component of the tension in each cord to be the same; for we have

$$t_1 \sin.\alpha_1 = t_2 \sin.\beta_1 = t_2 \sin.\alpha_2 = t_3 \sin.\beta_2 = \&c. \;.\;.\;. = t_n \sin.\beta_{n-1}$$

$$\text{but } t_1 \sin.\alpha_1 = t_1 \cos.A = t_n \sin.\beta_{n-1} = t_n \cos.F$$

or the resolved parts of the extreme tensions in the line AF are equal and opposite.

From the equations (1), (2), . . . $(n-1)$, we find by eliminating alternately one of the tensions in each pair,

$$\frac{t_1}{\sin.\beta_1} = \frac{W_1}{\sin.(\alpha_1 + \beta_1)} = \frac{t_2}{\sin.\alpha_1}$$

$$\frac{t_2}{\sin.\beta_2} = \frac{W_2}{\sin.(\alpha_2 + \beta_2)} = \frac{t_3}{\sin.\alpha_2}$$

&c. &c. &c.

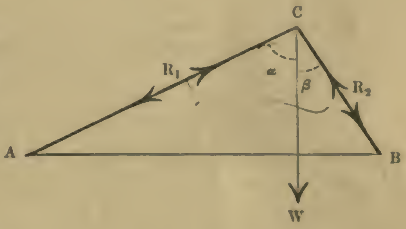
$$\frac{t_{n-1}}{\sin.\beta_{n-1}}=\frac{W_{n-1}}{\sin.(\alpha_{n-1}+\beta_{n-1})}=\frac{t_n}{\sin.\alpha_{n-1}}$$

From these we find the horizontal tensions to be

$$\frac{W_1}{\cot.\alpha_1+\cot.\beta_1}=\frac{W_2}{\cot.\alpha_2+\cot.\beta_2}=\&\text{c.}\ldots=\frac{W_{n-1}}{\cot.\alpha_{n-1}+\cot.\beta_{n-1}}$$

When the weight of a chain supplying the place of the cords of the polygon is taken into account, the problem becomes one of considerable difficulty, and is connected with the construction of chain bridges.

76. Prop. *Three uniform beams connected together form a triangle, ABC, with AB horizontal, and have a weight, W, hung from C; to find the reaction in each of the beams, and to shew that the horizontal part is the same in each beam.*



Since the triangle is given, the angles which the sides *AC*, *BC* make with the vertical direction will be known: let them be α and β respectively.

Let R_1 be the reaction in the beam *AC*, R_2 that in *BC*; and let b_1 be the weight of *AC*, b_2 that of *BC*.

Since the beams are uniform, the center of gravity of each will be at its middle point, and we may consider half the weight of each to be at each end, so that the whole weight hanging from *C* will be $W+\frac{b_1+b_2}{2}=W'$ say.

Resolving horizontally and vertically at *C*, we have

$$R_1\sin.\alpha-R_2\sin.\beta=0 \qquad (1)$$

$$R_1\cos.\alpha+R_2\cos.\beta-W'=0 \qquad (2)$$

The equation (1) shews that the horizontal parts of the reactions are equal.

Eliminating alternately between the equations, we have

$$\frac{R_1}{\sin. \beta} = \frac{W'}{\sin. (\alpha+\beta)} = \frac{R_2}{\sin. \alpha}$$

and the horizontal part of R_1 or R_2, or, as it is called, the *horizontal thrust* when the beams form *a roof*, is equal to

$$\frac{W'}{\cot. \alpha + \cot. \beta}$$

which is the reaction in the horizontal beam AB, called *the tie-beam.*

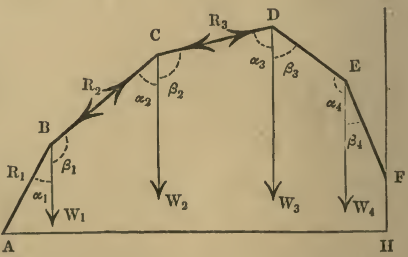
If the side AB were taken away, and its place supplied by a cord, the last expression would be the tension in the cord. If the points A and B rested on a smooth horizontal plane, we should have the pressure on the plane

$$\text{at } A = R_1 \cos. \alpha + \frac{b_1}{2}$$

$$\text{and at } B = R_2 \cos. \beta + \frac{b_2}{2}$$

77. Prop. *A number* (n) *of beams connected by hinges form a frame-work in a vertical plane, having weights hung from the hinges: to find the position of equilibrium, and to shew that the horizontal reaction at each hinge is the same.*

Let $ABCDEF$ be the frame-work, of which the extremities A and F are fixed points. Let AH and FH, a horizontal and vertical line, be given, as well as the lengths b_1, b_2, b_3, &c. of the beams.



The beams being given, we may suppose the weight to be applied at the ends, and W_1, W_2, W_3, &c. to be the whole weights acting at the hinges, B, C, D, &c. respectively.

Let α_1, β_1 be the angles which the beams AB, BC make with the vertical line at B; α_2, β_2, α_3, β_3, &c. the corresponding angles at C, D, &c.

Let R_1, R_2, R_3, &c. . . . R_n be the reactions in the beams commencing at A.

We shall have to find n reactions, and $2(n-1)$ angles. From the geometrical relations we have

$$AH-b_1\sin.\alpha_1-b_2\sin.\alpha_2-b_3\sin.\alpha_3-\&c.\ldots-b_n\sin.\beta_{n-1}=0 \quad (a)$$

$$FH-b_1\cos.\alpha_1-b_2\cos.\alpha_2-b_3\cos.\alpha_3-\&c.\ldots-b_n\times \cos.(\pi-\beta_{n-1})=0 \quad (b)$$

Resolving horizontally and vertically at each hinge, we have, in equilibrium,

$$\text{at } B,\ R_1\sin.\alpha_1-R_2\sin.\beta_1=0 \qquad R_1\cos.\alpha_1+R_2\cos.\beta_1-W_1=0 \quad (1)$$

$$\text{at } C,\ R_2\sin.\alpha_2-R_3\sin.\beta_2=0 \qquad R_2\cos.\alpha_2+R_3\cos.\beta_2-W_2=0 \quad (2)$$

&c. &c.

$$R_{n-1}\sin.\alpha_{n-1}-R_n\sin.\beta_{n-1}=0 \qquad R_{n-1}\cos.\alpha_{n-1}+R_n\cos.\beta_{n-1}-W_{n-1}=0 \quad (n-1)$$

We have also $\beta_1+\alpha_2=\pi$, $\beta_2+\alpha_3=\pi$, &c. . . . $\beta_{n-2}+\alpha_{n-1}=\pi$, or $n-2$ equations.

These equations being $3n-2$ in number suffice to find the n tensions and $2n-2$ angles.

The expressions would have been identical with those for the funicular polygon if we had taken F in the horizontal line through A; and they lead to the same consequences.

The first of each pair of equations (1), (2), &c. shews that the horizontal component of the reactions at each hinge is the same; and by the same process as in Article 75 we shew that it is the same at every hinge, and equal to

$$\frac{W_1}{\cot.\alpha_1+\cot.\beta_1}=\frac{W_2}{\cot.\alpha_2+\cot.\beta_2}=\&c.\ldots=\frac{W_{n-1}}{\cot.\alpha_{n-1}+\cot.\beta_{n-1}}$$

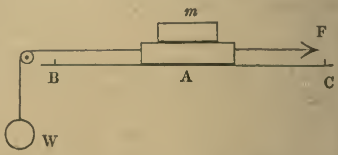
CHAPTER X.

ON FRICTION.

We have hitherto considered the surfaces on which bodies pressed to be perfectly smooth, so that they offered no resistance to motion parallel to themselves, their only reaction being perpendicular.

When rough surfaces are in contact, the motion, or tendency to motion, parallel to the surfaces, is affected by the roughness, and we call the effect friction.

Experiments have been made to determine the laws of friction, which we may subdivide into rubbing friction, when one body rubs on the other, and rolling friction, when one rough surface rolls upon another: the former only will be considered here, under the term *friction* or *statical friction.*

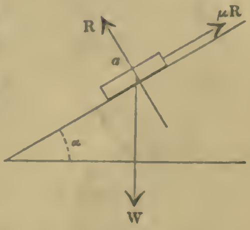


78. If a body rest, as at A, upon a rough plane, BC, it is found that a force, within certain limits, may act upon it parallel to the plane without motion ensuing, as would be the case if the plane were smooth. The greatest force which can be so applied, without the body moving, measures the friction. If W be a weight acting by a cord passing over a pulley, as in the figure, on the body A, when motion is about to take place, F being the opposing force of friction which balances W, we have $F=W$. It is found that if we put various weights, as m, upon the body A, then W or F is proportional to the weight of A and m, or is proportional to the pressure perpendicular to the surfaces in contact. It is also found that it is independent of the magnitude of the surfaces in contact, the friction being the same when the pressure is the same, whether the surface of the body be increased or diminished; except in extreme cases, where the

pressure is exceedingly great compared with the surfaces in contact. So that if R be the pressure on the plane which is equal and opposite to its normal reaction, we have

$$F = \mu R$$

where μ is a constant, depending on the nature of the surfaces in contact, and called *the coefficient of friction.*



We may determine the coefficient of friction by placing the body on a plane of which we can increase the inclination to the horizon until the body begins to slide down.

79. Prop. *To shew that the coefficient of friction between two given substances is the tangent of the inclination of the plane formed of one of the substances, when the body formed of the other is about to slide down it.*

The body a in the above figure is in equilibrium from the normal reaction of the plane R, the friction μR acting up the plane, and its weight acting vertically. Resolving parallel and perpendicular to the plane, we have

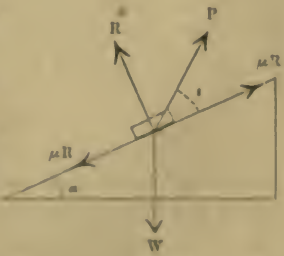
$$\mu R - W \sin. \alpha = 0$$

$$R - W \cos. \alpha = 0$$

Eliminating R, we have

$$\mu \cos. \alpha - \sin. \alpha = 0$$

$$\text{or, } \mu = \tan. \alpha$$



80. Prop. *To find the limits of the ratio of P to W on an inclined plane, when friction acts up or down the plane.*

Let the power P, as in the figure, make an angle ϵ with the plane whose inclination to the horizon is α. Let W be the weight of the body.

Friction being considered as an inert force resisting the tendency to motion, will act *up* or *down* the plane as the body is on the point of moving *down* or *up* respectively.

Resolving parallel and perpendicular to the plane, we have

$$P \cos. \epsilon \pm \mu R - W \sin. \alpha = 0 \qquad (1)$$

$$P \sin. \epsilon + R - W \cos. \alpha = 0 \qquad (2)$$

Multiply (2) by μ and subtract and add, we have

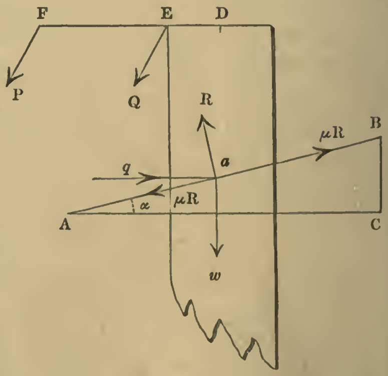
$$P(\cos. \epsilon \mp \mu \sin. \epsilon) - W(\sin. \alpha \mp \mu \cos. \alpha) = 0$$

$$\text{or, } \frac{W}{P} = \frac{\cos. \epsilon \mp \mu \sin. \epsilon}{\sin. \alpha \mp \mu \cos. \alpha}$$

where the upper sign is to be taken when friction acts up, and the lower when friction acts down the plane.

No motion will occur whilst the relation of P to W lies between the two values.

81. Prop. *To find the limits of the ratio of P to W in the screw when friction acts assisting the power or the weight.*



Proceeding as in Article 54, let ABC be the inclined plane formed by the unwrapping of one revolution of the thread; the angle $BAC = \alpha$.

Let W be the whole weight sustained by the screw; w be the part of it supported at a; Q the whole force acting at the circumference of the cylinder, whose radius $ED = r$, $FD = a$ the lever at which the power P acts, and

$$\therefore P\frac{a}{r} = Q$$

and let q be the part of Q which supports w at a. The forces which are in equilibrium at a are the weight w, the reaction R, the friction acting *up* or *down* the plane μR, and the horizontal pressure q.

Resolving parallel and perpendicular to the plane, we have

$$q\cos.\alpha \pm \mu R - w\sin.\alpha = 0 \qquad (1)$$

$$q\sin.\alpha - R + w\cos.\alpha = 0 \qquad (2)$$

Multiply (2) by μ, and add and subtract, we have

$$q(\cos.\alpha \pm \mu\sin.\alpha) - w(\sin.\alpha \mp \mu\cos.\alpha) = 0$$

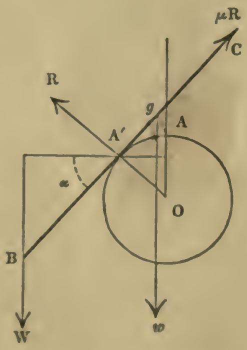
$$\text{or, } \frac{w}{q} = \frac{W}{Q} = \frac{\cos.\alpha \pm \mu\sin.\alpha}{\sin.\alpha \mp \mu\cos.\alpha}$$

$$\text{and } \frac{W}{P} = \frac{a}{r}\cdot\frac{\cos.\alpha \pm \mu\sin.\alpha}{\sin.\alpha \mp \mu\cos.\alpha}$$

The two values of this expression give the limits required.

82. **Prob.** *A uniform beam rests on a cylinder of given radius* (a)*; to find the weight* (W) *which may be hung from one end, so that the beam may be just about to slide off when friction acts.*

Let g be the center of gravity of the beam BC, whose length is $2b$; and $Bg=b$, since the beam is uniform. Let w be its weight. Before the weight W was hung at B, the point g must have been at A, the highest point of the cylinder. Let A' be the point of contact when the beam is on the point of sliding off the cylinder, and let α be the angle which the beam then makes with the horizontal direction.



Resolving parallel and perpendicular to the beam, we have

$$\mu R - w\sin.\alpha - W\sin.\alpha = 0 \qquad (1)$$

$$R - w\cos.\alpha - W\cos.\alpha = 0 \qquad (2)$$

whence $\mu = \tan.\alpha$.

Taking the moments about A', we have

$$W \times BA'\cos.\alpha - w \times A'g\cos.\alpha = 0$$

$$\text{or, } W(Bg - A'g) - w.A'g = 0$$

$$\text{but } A'g = \text{arc } AA' = \text{radius} \times \text{angle } AOA'$$

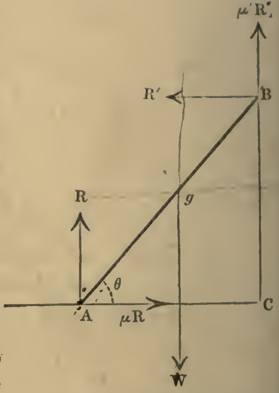
$$= a\alpha$$

$$\therefore W(b-a\alpha)-w\,.\,a\alpha=0$$

$$\text{or, } W=\frac{w\,.\,a\alpha}{b-a\alpha}=\frac{wa\tan.^{-1}\mu}{b-a\tan.^{-1}\mu}$$

the weight required.

83. Prob. *A ladder rests with its foot on a horizontal plane, and its upper extremity against a vertical wall; having given its length* (l), *the place of its center of gravity, and the ratios of the friction to the pressure both on the plane and on the wall; find its position when in a state bordering on motion.*



If AB be the ladder in the figure, whose length $AB=l$, and weight acting at the center of gravity $g=W$, $Bg=b$, μ' the coefficient of friction against the vertical wall, μ against the horizontal plane AC, and angle $BAC=\theta$; we find

$$\tan.\,\theta=\frac{l-b\,(1+\mu\,\mu')}{\mu\,l}$$

DYNAMICS.

CHAPTER I.

ON DEFINITIONS AND THE LAWS OF MOTION.

When forces produce motion, or change of motion, in bodies, their effect being different to that in statical problems, we require other methods of measuring them in addition to the statical measures.

The *motion of a body* may be considered only with respect to its change of place, either *absolutely*, or *relatively* to some other body which is itself in motion. It may also be considered with respect to the power the body acquires of overcoming obstacles, and then the magnitude of the body itself has to be considered.

The *velocity* of a body is its rate of motion, and it is measured, when *uniform* or *constant*, by the space passed over in a unit of time, or by the space in any time divided by the time. The units of time and space must be known. Thus we say, he travelled at the rate of thirty miles per day; he travelled at the speed or velocity of eight miles per hour; the bullet was fired from the gun with a velocity of 1000 feet per second. When no mention is made of different units, we shall take a foot for the unit of space, and a second for the unit of time.

Let v be put for velocity, s for space in feet, t for time in seconds, we have for a uniform velocity,

$$\begin{aligned} v &= \text{space described in one second} \\ &= \frac{\text{space described in } t \text{ seconds}}{t \text{ seconds}} \\ &= \frac{s}{t}, \quad \text{or } s = vt \end{aligned}$$

When the velocity is continually changing, it is measured by the space passed over in an indefinitely short space of time divided by the time.

Let s be the space described with a variable velocity in t seconds, s' that in t' seconds indefinitely near the former time; then, using the symbol δ to signify difference,

$$v = \frac{s' - s}{t' - t} = \frac{\delta s}{\delta t}$$

When *pressures* such as we have considered in Statics are not balanced, the body on which they act will be put in motion. Such pressures may act on the body during a definite time, or act through a definite space, and then cease to act.

When pressures act only for a very short space of time, they are called *impulsive* forces; as, for instance, the mutual pressure of bodies which impinge, the force of the string exerted upon an arrow shot from a bow, &c.

We call a force an *accelerating force* whilst it continually increases the velocity of a body; and a *retarding force* whilst it continually diminishes it. A *uniform* or constant *accelerating* force is one that increases the velocity of the body uniformly, or adds the same amount of velocity to the previous velocity of the body in every successive equal interval of time. A *uniform retarding* force is one that diminishes the velocity of a body according to the same law. *Variable* accelerating or retarding forces are those whose effects on the velocities of bodies are continually changing.

By the *moving force* acting on a body, we mean the *mass moved* multiplied by the *accelerating force*, which we shall shortly see, by the third law of motion, is proportional to the pressure exerted on the body, by whatever means that pressure arises, whether by the unbending of a spring, by the attraction of other bodies upon it, by the explosion of gunpowder, &c.

By the *momentum* of a moving body, we mean the mass of the body multiplied by its velocity. The mass multiplied into the square of the velocity is called the *vis viva* of a moving body.

As forces are measured by the effects they produce, the dynamical measure of an accelerating force must be the velocity which it generates in the body in a given time, when uniform or constant. Let f represent the force, and let the given time be taken as the unit, we have $f=$the velocity generated in a unit of time. Since the force is constant, the same velocity is added in every unit of time; therefore, if v be the velocity acquired from rest by the action of the force in t units of time, we have $v=ft$. From the above definitions we see that *moving force* is measured by the momentum generated in a unit of time.

From the expression $v=ft$, we have

$$f=\frac{v}{t}=\frac{\text{velocity generated in any time}}{\text{time}}$$

When the force is *variable*, and changing from one instant to another, we must take the time indefinitely small; so that, for a variable force, we have

$$f=\frac{\text{velocity generated in an indefinitely small time}}{\text{time}}$$
$$=\frac{v'-v}{t'-t}=\frac{\delta v}{\delta t}$$

if $v=$ velocity of the body at the time t, and v' the velocity at the time t' indefinitely near to t.

In the above, it will be shortly seen that we have anticipated the first law of motion.

By the *path* of a body we mean the line, straight or curved, which it describes in passing from one point to another in space.

The pressure produced by the weight of a body depends upon the attraction of gravitation towards the earth, which is sensibly different at different parts of the earth, and upon the mass of the body; and varies directly as the force of gravity when the mass is the same, and directly as the mass when the force of gravity is the same; therefore, by the rules of algebra, when both vary, the weight varies as their product. Let m be the mass, w the weight of a body, and g the force of gravity: we have $w \propto mg$, and taking our units of measure of w, g, and m accordingly, we may put $w = mg$, or $m = \frac{w}{g}$.

The *relative* velocity of two bodies is the velocity with which they approach each other, or separate from each other.

A body is said to move *freely* when its path depends on the action of the impressed forces only. Its motion is said to be *constrained* when its path is limited to be in a given line, straight or curved, or to be upon a given surface. A stone thrown in any direction from the hand is an example of the former; and a pendulum swinging, or a body rolling down a hill, are examples of the latter.

ON THE THREE LAWS OF MOTION.

The first law of motion. When a body in motion is not acted on by any external force, it will move in a straight line, and with a uniform velocity.

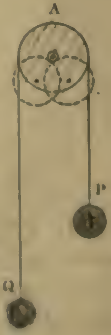
We may satisfy ourselves of the truth of the first part of the law, that the body not acted on by any external force will describe a straight line, from the consideration that whatever reason could be alleged for its deviating to one side, as good a reason could be given for its deviating to the opposite; and since it could not move in two directions at the same time, the reasons could not be valid, and therefore it would move in a straight line only.

The second part of the law we conclude to be true by induction from the results of our experience. If a body be thrown along a rough surface, it deviates from a straight path, and soon loses all its velocity; if it be thrown along a smoother level surface, it moves in a path nearer a straight line, and with a velocity more slowly diminished; if it be thrown along a sheet of ice, it moves very nearly in a straight line, and retains its motion for a considerable time. If a heavy ball be suspended by a very fine thread in a vessel from which the air has been withdrawn by the air-pump, and it is set oscillating, it remains in motion for a very great length of time, although it still experiences resistance from the remaining air in the vessel, and the want of perfect flexibility in the thread. We therefore conclude, that if we could remove all the resistance to a body in motion, it would retain its velocity unchanged.

The second law of motion. If several forces act upon a body at the same time, each produces its full effect in the direction of its action, whether the body be at rest or in motion.

We see this to be true when a ball is rolled along the deck of a vessel moving uniformly in smooth water: the ball takes the same course along the deck as it would if the vessel were at rest; and if it struck any body in its motion, the forces called into play in the collision would produce the same effects in the two cases. If a body be let fall from the top of the mast, it falls to the foot of the mast, whether the vessel be at rest or in motion. A more complete proof is afforded by experiments with the pendulum. Whatever be the vertical plane in which it oscillates, at the same place on the earth's surface, whether north and south, east and west, or in any other azimuth, the time of oscillation is the same; shewing that the effect of gravity on the pendulum is unaffected by the rotation of the earth on its axis, and by its motion in its orbit.

The third law of motion. When a pressure acting on a body puts it in motion, the moving force (measured by the momentum generated in a unit of time) is proportional to the pressure.



This law is proved experimentally by Atwood's machine, which consists of a pulley, *A*, having its axle resting on two friction-wheels at each end, as represented by the dotted circles. These friction-wheels have their pivots, or axles, accurately and delicately fixed, so that the pulley *A* may be subject to as little effect of friction as possible. A clock which goes for a few minutes, with a second's pendulum and deadbeat escapement, is connected with the frame-work supporting the machine.

Let P and Q be the weights of the bodies suspended from the ends of the cord passing over the pulley A. If $P=Q$, they will balance; but if one (P) be greater than the other (Q), it will descend and draw the other upwards, with a force $=P-Q$, which is the pressure setting in motion the masses of P, Q, and of the wheel-work A. The masses of P and Q are $\frac{P}{g}$ and $\frac{Q}{g}$ respectively. Let I represent the inertia or mass of the wheel-work, acting upon the cord connecting P and Q.

The whole mass set in motion $=\frac{P+Q}{g}+I$, and the pressure producing motion $=P-Q$. Now, it is found that when these two quantities are kept in the same proportion to each other, the velocity acquired in a unit of time is the same; or the pressure varies as the mass moved, when the velocity is constant.

If part of Q be taken successively away, and added to P, so that the mass moved remains the same, it is found that the velocity acquired in a unit of time is proportional to $P-Q$; or the pressure varies as the velocity acquired in a unit of time, when the mass moved is constant.

When both the mass and velocity change, by the rules of variation, we have generally,

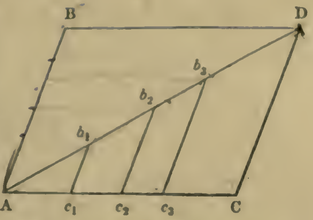
pressure varies as mass moved multipled by the velocity generated in a unit of time,

varies as mass multiplied into accelerating force;

varies as moving force.

ON THE PARALLELOGRAM OF VELOCITIES.

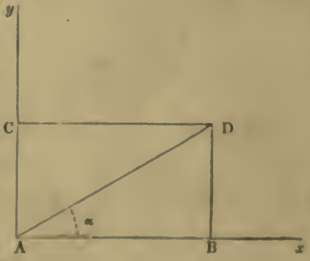
1. Prop. *If two velocities be impressed upon a body at the same instant, the actual velocity of the body will be represented in direction and magnitude by the diagonal of the parallelogram formed upon the lines representing the impressed velocities.*



Let a body at A have a velocity impressed upon it which would carry it with a uniform motion from A to B in a given time, and another velocity at the same instant which would carry it similarly from A to C in the same time. If we complete the parallelogram $ABDC$, the actual path of the body will be the diagonal AD, described in the same given time.

By the *second law of motion* each velocity is impressed independently of the other, and if the body had passed over the spaces Ac_1, Ac_2, Ac_3 in any times in the direction of AC, it would have passed over the spaces c_1b_1, c_2b_2, c_3b_3 parallel to AB, in the same times respectively; so that the latter spaces bear to the former respectively the same ratio that AB bears to AC, and therefore the points b_1, b_2, b_3 will be all in the straight line AD, which is the diagonal of the parallelogram. The body will also have arrived at D in the same time as it would have passed over either of the spaces AB or AC.

2. PROP. *Having given a velocity, to find the component velocities in any directions at right angles to each other; and having given two component velocities at right angles to each other, to find the resultant velocity of a body.*



Let Ax, Ay be the directions, at right angles, in which the components are required. Let AD represent the velocity (v) of the body in direction and magnitude, α the angle which AD makes with Ax. If we complete the right-angled parallelogram $ABDC$, the side AB will represent the velocity (a) in Ax, and AC (b) that in Ay.

Then $AB = AD \cos. \alpha$ $\qquad$ $AC = AD \sin. \alpha$

or, $a = v \cos. \alpha$ $\qquad$ $b = v \sin. \alpha$

Again, if the components a and b in Ax, Ay are given to find the magnitude and direction of the resultant velocity v, we have

$$v^2 = a^2 + b^2$$

$$\text{and tan. } \alpha = \frac{b}{a} \text{ as required.}$$

We have seen in Statics that, if two bodies are in equilibrium from their mutual pressures, their *actions* upon each other must be equal and opposite; so also with two bodies in which motion takes place, the mutual pressures or actions are equal and opposite. By the third law of motion, the moving force is proportional to the pressure; and hence when two bodies move by the effect of their mutual actions, the moving force produced in each of the bodies is the same in magnitude, but opposite in direction.

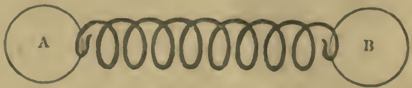
This consequence of the third law of motion is called the principle, *that action and reaction are equal and opposite*, each being measured by the momentum generated in a given time, the effect being considered uniform during that time.

CHAPTER II.

ON THE IMPACT OR COLLISION OF BODIES.

When two bodies in motion impinge, they exert a mutual but varying pressure, during an interval of time which is generally very short. The forces called into play are subject to the principle *that action and reaction are equal and opposite;* and since this is true at each instant of the mutual pressures, the whole effects of the impulsive forces will be subject to the same principle.

When it is the final and *completed* result that we require, we have only to consider the mutual pressures in the collision to have produced their full effect and to have ceased; we can, however, find the circumstances of the motion of the bodies during the short interval of mutual pressure, by employing a higher analysis than can be admitted in this treatise. For example, if the mutual pressures of the bodies *A* and *B* in the figure acted by a spiral spring of known elasticity, the circumstances of the motion of each body during the compression of the spring are easily determined.



When natural bodies impinge, we have a similar case to the one just taken for example.

If two surfaces of india-rubber be pressed against each other, we see them flattened by the pressure, but recover their former shape as the pressure is removed. Although not so perceptible, the same takes place in other elastic bodies.

During the impact of two elastic bodies, the force urging them towards each other is called the force of *compression;* and the opposing force, causing them to separate again, is called the force of *restitution.* The ratio which the force of *restitution* bears to the force of *compression* in any bodies of the same material or substance, is *nearly* the same for all degrees of compression, and measures the *elasticity* of the substance. The value of this ratio is called the *modulus of elasticity*.

The whole force of compression is measured by the momentum destroyed during the approach, and the whole force of restitution by that generated during rebounding; and the momentum destroyed or generated in one of the bodies equals that destroyed or generated in the other, because action and reaction are equal and opposite.

Let ϵ be the modulus of elasticity of two bodies whose masses are A and B, which, moving with their centers of gravity in the same straight line, meet directly with the opposite velocities a and b, and rebound with the opposite velocities a' and b' respectively: let v be the velocity supposed in the direction of a, common to both bodies at the instant when compression ends and rebounding commences; we have

$$\text{modulus of elasticity} = \frac{\text{whole force of restitution}}{\text{whole force of compression}}$$

$$= \frac{\text{momentum generated in either } A \text{ or } B \text{ in rebounding}}{\text{momentum destroyed in either } A \text{ or } B \text{ during compression}}$$

which gives us

$$\epsilon = \frac{A(a'+v)}{A(a-v)} \qquad \text{or, } \epsilon a - \epsilon v = a' + v$$

$$\text{and } \epsilon = \frac{B(b'-v)}{B(b+v)} \qquad \text{or, } \epsilon b + \epsilon v = b' - v$$

adding, we eliminate v, and have

$$\epsilon = \frac{a'+b'}{a+b}$$

$$= \frac{\text{velocity of separation}}{\text{velocity of approach}}$$

so that when experiments are tried in which the velocities of rebounding can be determined for any given velocities of impact, the above ratio gives the modulus of elasticity.

Bodies suspended by fine cords, and allowed to oscillate in circular arcs of given radius about a fixed point as center, acquire, as will be shewn in the chapter on constrained motion, velocities at the lowest point which are proportional to the chords of the arcs fallen through, and similarly they ascend through arcs of which the chords are proportional to the velocities impressed upon them at the lowest point of the arcs. They are also shewn to fall down all *small* arcs of the same circle in the same time. If two bodies be suspended in this manner, so as to impinge at the lowest points of the arcs they describe, and be let fall simultaneously from given points, we know the velocities with which they impinge, and by observing the arcs through which they ascend after rebounding, we know the velocities of rebounding, and so have sufficient data to determine the modulus of elasticity.

All known solid bodies are *imperfectly* elastic—that is, in all, the force of *restitution* is less than the force of compression; but we have none without some force of restitution, or which are perfectly non-elastic. Experiments to prove the mathematical results for the impact of non-elastic bodies require to be tried with balls of soft clay recently worked up, so that they adhere together after impact, or balls of wood with metallic spikes fixed in one of them, which entering the other ball prevent them separating again after impact.

We are indebted to the excellent experiments of Mr. Eaton Hodgkinson for our more accurate knowledge of the properties of impinging bodies. The following table of moduli, and the rule for bodies of different hardnesses, are from his results:

	Modulus of Elasticity.
Perfect elasticity	1·
Glass	·94
Hard-baked clay	·89
Ivory	·81
Limestone	·79
Steel (hardened)	·79
Cast-iron	·73
Steel (soft)	·67
Bell-metal	·67
Cork	·65
Elm-wood, across the fibres . . .	·60
Brass	·41
Lead	·20
Clay, just malleable by the hand .	·17

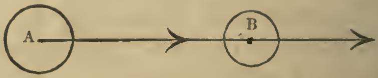
In the impacts between bodies whose hardness differs in any degree, the resulting elasticity is made up of the elasticities of both, according to the following formula:

$$\text{modulus of elasticity from both} = \frac{h'\epsilon + h\epsilon'}{h+h'}$$

where h and h' are the relative hardnesses, and ϵ and ϵ' the moduli of elasticity respectively, of the bodies.

From this rule, it results, that if one of the bodies is much harder than the other, the effective elasticity is that of the softer body nearly.

3. Prop. *To find the common velocity of two non-elastic spherical bodies after direct impact.*



Let A and B be the masses of the bodies as in the figure, moving in the line joining their centers as indicated by the arrows, so that they impinge *directly*, and not *obliquely*, when A overtakes B.

Let a be the velocity of A, b of B, and a greater than b. Let v be the velocity after impact, which is to be found. The velocity lost by A is $(a-v)$, and that gained by B is $(v-b)$.

Since *action* and *reaction* are equal and opposite, we have the momentum lost by A equal to the momentum gained by B, or

$$A(a-v) = B(v-b)$$

$$\text{whence } v = \frac{Aa+Bb}{A+B}$$

When the bodies are moving in opposite directions, we must take the velocities with contrary signs. Let B's velocity be $-b$, we have

$$v = \frac{Aa-Bb}{A+B}$$

If $Aa=Bb$, then $v=0$, or the bodies remain at rest after impact. If Bb is greater than Aa, then v is negative, or in the direction of B's motion.

4. Prop. *The velocities of two imperfectly elastic spherical bodies of the same substance which impinge directly being given, it is required to find their velocities after impact.*

Referring to the figure of the last Prop., let A and B be the masses of the impinging bodies A and B; let a, b be their velocities respectively before impact; a', b' those after impact, which are to be found.

Then, if ϵ be the modulus of elasticity of the bodies, we have

$$\epsilon=\frac{\text{velocity of separation}}{\text{velocity of approach}}$$

$$=\frac{b'-a'}{a-b} \qquad (1)$$

$$\text{and momentum lost by } A=A(a-a')$$

$$\text{momentum gained by } B=B(b'-b)$$

Since *action* and *reaction* are equal and opposite, we have

$$A(a-a')=B(b'-b) \qquad (2)$$

Substituting in (2) the values of b' and a' successively from (1), we have

$$a'=\frac{Aa+Bb}{A+B}-\frac{B\epsilon(a-b)}{A+B}$$

$$b'=\frac{Aa+Bb}{A+B}+\frac{A\epsilon(a-b)}{A+B}$$

If B be moving in the opposite direction to A, and we consequently take its velocity negative and equal to $-b$, the expressions become

$$a'=\frac{Aa-Bb}{A+B}-\frac{B\epsilon(a+b)}{A+B}$$

$$b'=\frac{Aa-Bb}{A+B}+\frac{A\epsilon(a+b)}{A+B}$$

If B were at rest, or $b=0$, the expressions become

$$a'=\frac{a(A-B\epsilon)}{A+B}$$

$$b'=\frac{Aa(1+\epsilon)}{A+B}$$

We see that b' can never in this case $=0$, but a' will $=0$ if

$$\frac{A}{B}=\epsilon$$

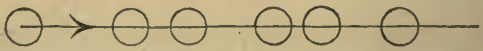
If in the preceding expressions we put $\epsilon=1$, the results will be those for perfectly elastic bodies.

If we take $A=B$, and $\epsilon=1$, when the bodies meet, moving in opposite directions with any velocities a and b respectively, we have the expressions becoming

$$a'=\tfrac{1}{2}(a-b)-\tfrac{1}{2}(a+b)=-b$$
$$b'=\tfrac{1}{2}(a-b)+\tfrac{1}{2}(a+b)=a$$

or, in direct impact the balls exchange velocities.

If we have a series of n perfectly elastic and equal balls arranged in one straight line, and the ball at one extremity be projected against the next, each ball will remain at rest after striking the succeeding one until we come to the last, which will fly off, having the velocity with which the first was projected. It is immaterial at what distances the balls be placed; and if in contact, the impinging of the first ball appears to produce no *visible* effect but causing the last one to fly off with its velocity of impact, the others remaining stationary. The same holds good for imperfectly elastic balls, if each bears to the one it strikes the ratio $\epsilon:1$, as we see above in the case of $\frac{A}{B}=\epsilon$, when $a'=0$ whatever a may be, when $b=0$; also $b'=a\epsilon$, or the velocity of each ball is less than that of the preceding one.



5. Prop. *If any two spherical bodies of the same substance impinge directly, the motion of their center of gravity after impact is the same as before impact.*

Since the bodies impinge directly, their common center of gravity, both before and after impact, will be always in the line joining their separate centers of gravity; or its motion will be always in this line.

Let u be the velocity of the common center of gravity of the bodies A and B, moving in the same direction with velocities a and b respectively, before impact.

Let x_1, x_2, $\bar{x}$ be the distances of the centers of A and B, and of their center of gravity from any fixed point in their line of motion, at any instant; x_1', x_2', $\bar{x}'$ the same quantities after an interval of time t, so that

$$x_1' = x_1 + a\,t$$
$$x_2' = x_2 + b\,t$$
$$\bar{x}' = \bar{x} + u\,t$$

By the properties of the center of gravity (Articles 34 and 16 in Statics), we have

$$(A+B)\bar{x} = Ax_1 + Bx_2 \qquad (1)$$
$$(A+B)\bar{x}' = Ax_1' + Bx_2'$$

or substituting for $\bar{x}'$, x_1', x_2', the values above, the latter equation becomes

$$(A+B)(\bar{x}+ut) = A(x_1+at) + B(x_2+bt)$$

Subtracting (1) from this, we have

$$u = \frac{Aa+Bb}{A+B}$$

which is the same as the velocity of the center of gravity of non-elastic bodies after impact.

Let u' be the velocity of the center of gravity of the bodies after impact, when imperfectly elastic; we have similarly,

$$u' = \frac{Aa'+Bb'}{A+B}$$

Substituting the values of a' and b' from Article 4, we have

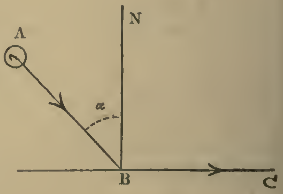
$$u' = \frac{A}{A+B}\left(\frac{Aa+Bb}{A+B} - \frac{B\epsilon(a-b)}{A+B}\right) + \frac{B}{A+B}\left(\frac{Aa+Bb}{A+B} + \frac{A\epsilon(a-b)}{A+B}\right)$$
$$= \frac{Aa+Bb}{A+B}$$
$$= u$$

or the velocity of the center of gravity is unchanged by impact.

When a body impinges against a *fixed* plane surface, we can determine its motion after impact, if the direction of the motion and velocity before impact be given, together with the modulus of elasticity between it and the plane.

The angle which the direction of the motion before impact makes with the *perpendicular* to the surface at the point of impact is called the *angle of incidence;* and the angle which the direction of the motion after impact makes with the same line is called the *angle of rebounding* or *of reflexion.*

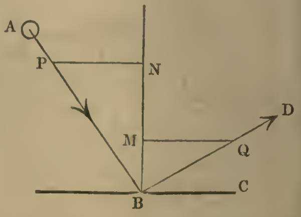
6. PROP. *If a smooth non-elastic body impinge on a smooth, non-elastic, hard, and fixed plane, it will, after impact, slide along the plane.*



Let the body A impinge upon the plane BC at B, making an angle α with BN, the perpendicular to the plane at B; let the velocity of A be v.

The component of v parallel to the plane $= v \sin. \alpha$; this will not be changed by the impact, since the ball and the plane are smooth. The velocity perpendicular to the plane after impact will be *nothing,* since there is no elasticity; therefore the body after impact will slide along the plane with the velocity $v \sin. \alpha$.

7. PROP. *To determine the velocity and direction of motion after impact, when an imperfectly elastic spherical body impinges obliquely on a smooth, hard, and fixed plane.*



Let the body A in the figure impinge on the plane BC at B, making the angle of incidence $ABN = \alpha$; let $\alpha' =$ the angle of reflexion NBD.

Let the line PB represent v the velocity of the body before impact; draw PN parallel to BC, then PN represents the velocity of the body parallel to the plane $= v \sin. \alpha$, and BN represents the velocity perpendicular to the plane $= v \cos. \alpha$. If ϵ be the modulus of elasticity between the body and the plane, the velocity perpendicular to the plane after impact will be $\epsilon v \cos. \alpha = MB$, if we take $\frac{MB}{NB} = \epsilon$.

From M measure $MQ = NP$ the parallel velocity, which is unchanged in impact, since the plane and body are smooth; BQD is the direction of the motion after impact, and the line BQ represents the velocity. Now

$$BQ^2 = MQ^2 + BM^2$$
$$= v^2 \sin.^2 \alpha + \epsilon^2 v^2 \cos.^2 \alpha$$

Or if u be the velocity after impact, we have

$$u = v \sqrt{\sin.^2 \alpha + \epsilon^2 \cos.^2 \alpha}$$

Also,

$$\tan. \alpha' = \frac{MQ}{MB} = \frac{PN}{\epsilon . NB}$$
$$= \frac{\tan. \alpha}{\epsilon}$$

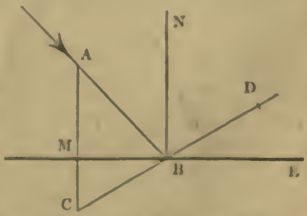
which give the direction of the motion and the velocity after impact.

If the elasticity be perfect, or $\epsilon = 1$, we have

$$\alpha' = \alpha, \text{ and } u = v$$

8. **Prop.** *To find the path of a body which, having passed through one given point, after rebounding from a given plane, passes through another given point.*

Let MBE be the given plane; A the point through which the body is projected; D the point through which it passes after rebounding from the plane.



Draw a perpendicular AMC to the plane through A, and if ϵ be the modulus of elasticity between the body and the plane, take $\frac{MC}{AM} = \epsilon$. Draw through the points C and D the line CBD, cutting the plane in B. Join AB; then AB, BD is the path required, or the body projected from A in the direction AB will rebound in the direction BD.

For the angle of reflexion DBN = angle BCM, and angle of incidence ABN = angle BAM,

$$
\begin{aligned}
\therefore \tan. DBN &= \tan. BCM \\
&= \frac{BM}{MC} \\
&= \frac{BM}{\epsilon . MA} \\
&= \frac{\tan. ABN}{\epsilon}
\end{aligned}
$$

or the angles ANB and DBN have the required relation as found in the last proposition.

EXAMPLES IN IMPACTS.

Ex. 1. Two bodies A and B, whose elasticity is m, moving in opposite directions with velocities a and b, impinge directly upon each other; shew that their distance at a time t after impact is $tm(a+b)$.

Ex. 2. If two perfectly elastic balls, whose masses are in the ratio 1 : 3, meet directly, with equal velocities; shew that the larger one remains at rest after impact.

Ex. 3. When two perfectly elastic bodies impinge directly, shew that the sum of the *vires vivæ* after impact equals their sum before impact: or, prove the expression

$$Aa^2 + Bb^2 = Aa'^2 + Bb'^2$$

Ex. 4. When two perfectly non~~in~~elastic bodies, A and B, moving in the same direction, impinge, shew that

$A+B : A ::$ relative velocity before impact : velocity gained by B.

Ex. 5. An imperfectly elastic ball is projected from a point in the circumference of a circle, and after twice rebounding from the circle returns to the same point again; shew that the direction of projection makes an angle α with the radius drawn to the point of projection which is given by the equation

$$\tan. \alpha = \frac{\epsilon^{\frac{3}{2}}}{\sqrt{1+\epsilon+\epsilon^2}}$$

CHAPTER III.

ON UNIFORM ACCELERATING FORCES AND GRAVITY.

ACCORDING to the definition of a uniform accelerating force in Chapter I., the velocity generated by it in the same time is always the same, and by the second law of motion, is unaffected by the previous motion of the body. If we put $f =$ the force, measured by the velocity it generates in a unit of time, we shall have the velocity generated in t units $= ft$. Writing v for this velocity acquired by the body at the end of the time t from rest, we have therefore

$$v = ft$$

9. PROP. *To find the relations of the* space, time, *and* force *when a body moves from rest under the action of a uniform accelerating force.*

The velocity of the body is continually increased from 0 up to ft, if t be the time and f the force. Let s be the whole space described in the time t, and let t be divided into n equal intervals, each $= \frac{t}{n}$. The velocities at the end of the times

$$\frac{t}{n},\ \frac{2t}{n},\ \frac{3t}{n},\ \frac{4t}{n},\ \text{\&c.} \ldots \frac{(n-1)t}{n},\ t,$$

will be respectively,

$$f\frac{t}{n},\ f\frac{2t}{n},\ f\frac{3t}{n},\ f\frac{4t}{n},\ \text{\&c.} \ldots f\frac{(n-1)t}{n},\ f\frac{nt}{n}$$

Now, if the body moved *uniformly* during each interval of time with the velocity it had at the *beginning* of the interval, from the expression space = velocity × time, we should have the whole space s equal to the sum of this series:

$$0+f\frac{t^2}{n^2}+f\frac{2t^2}{n^2}+f\frac{3t^2}{n^2}+\&c. \; . \; . \; . \; +f\frac{(n-1)t^2}{n^2}$$

$$=f\frac{t^2}{n^2}\{1+2+3+\&c. \; . \; . \; . \; +(n-1)\}$$

$$=f\frac{t^2}{n^2}\times\frac{n(n-1)}{2}$$

$$=\frac{ft^2}{2}-\frac{ft^2}{2n}$$

If the body had moved *uniformly* during each interval of time with the velocity it had at the end of the interval, we should have s equal the sum of this series:

$$f\frac{t^2}{n^2}+f\frac{2t^2}{n^2}+f\frac{3t^2}{n^2}+\&c. \; . \; . \; . \; +f\frac{(n-1)t^2}{n^2}+f\frac{nt^2}{n^2}$$

$$=f\frac{t^2}{n^2}(1+2+3+\&c. \; . \; . \; . \; +n)$$

$$=f\frac{t^2}{n^2}\cdot\frac{(n+1)n}{2}$$

$$=f\frac{t^2}{2}+f\frac{t^2}{2n}$$

Since the velocity is continually accelerated, the true value of s will be between these two quantities, however small each interval may be, or however great n may be; but when n is indefinitely great, the last terms in each of the above expressions vanish, and we have therefore

$$s=\tfrac{1}{2}ft^2$$

10. Between the two equations $v=ft$, and $s=\frac{1}{2}ft^2$, we may eliminate either f or t, and thus obtain

$$v^2=2fs \qquad\qquad s=\tfrac{1}{2}vt$$

The expression $s=\frac{1}{2}vt$ shews us that the space described from rest by the action of a uniform accelerating force is one half of the space which would have been described in the same time if the velocity had been constant and equal to its value at the end of the time.

If we put $t=1$ in the equation $s=\frac{1}{2}ft^2$, we have $f=2s$; or,

f, which is the velocity generated by the force in a unit of time, is measured by twice the space through which the body falls in a unit of time. It is found that a heavy body in our latitudes falls through a space of nearly 16·1 feet in the first second of time; therefore, if we put $g =$ the accelerating force of gravity, we have $g=$velocity of 32·2 feet per second of time, or, with the understanding that one second is our unit of time, we write $g=32{\cdot}2$ feet.

This value of the force of gravity is only an approximate value for small heights above the earth's surface.

Sir Isaac Newton's law of universal gravitation is, that every particle of matter attracts every other particle with a force which varies directly as the mass of the attracting particle, and inversely as the square of the distance. It is also shewn that a spherical body equally dense at equal distances from its center attracts a particle *outside* its surface as if the matter of the sphere were collected at its center; so that, considering the earth such a sphere, if g be the force of gravitation at the surface, f the force at any point *exterior* to the surface at a distance r from the center, and R be earth's radius, we have

$$f : g :: \frac{1}{r^2} : \frac{1}{R^2}$$

$$\text{or, } f = g\frac{R^2}{r^2}$$

$$= \frac{\mu}{r^2} \qquad \text{if } \mu = gR^2$$

μ in this expression is called the absolute accelerating force, for it is the value of f when $r=1$. When R is taken for unity, r must be expressed in terms of the earth's radius, and $\mu=g=32{\cdot}2$ feet.

When r is very near R, or for small heights above the earth's surface, we have $f=g$ very nearly, or we may take gravity as a constant accelerating force in that case. It is found that, on account of the diurnal rotation of the earth on its axis, its figure differs sensibly from spherical, being flattened at the poles and bulging at the equator, or is an oblate spheroid. The centrifugal force (see Article 31), being produced by the diurnal rotation, is greatest at the equator, and is *there* directly opposed to the force of gravity. It is also nothing at the poles. The resultant gravitation of a heavy body is affected by the direct action of the centrifugal force and its indirect action through the change of the figure of the earth. The ratio for the equator and poles is as follows:

gravitation at the equator : gravitation at the pole : : 186 : 187.

We can, for these reasons, only consider gravity as constant for the same latitude on the earth's surface, and for small altitudes above it. The direction of gravity at each point on the earth's surface being perpendicular to the surface taken as that of *still water*, does not pass *accurately* through the center of the oblate spheroid.

11. Prop. *A body being projected with a given velocity u in the direction in which a uniform accelerating force f acts; to find its velocity, and the space passed over in a given time.*

If v be the velocity, s the space described at the end of the time t, we shall have, by the second law of motion,

$$v = \text{velocity of projection} \pm \text{velocity from the action of the force}$$
$$= u \pm ft$$

where the upper sign is to be taken when the force accelerates the velocity, and the lower when it retards it.

In the same way,

$$\text{space described} = \text{space due to velocity of projection} \pm \text{the space due to the action of the force}$$

$$\text{or, } s = ut \pm \tfrac{1}{2}ft^2$$

the upper and lower signs to be taken as before.

12. Prop. *A body being projected with a given velocity u in the direction in which a uniformly accelerating force f acts; to find its velocity when it has passed through a given space.*

Let v be the velocity when the body has passed through the space s.

Let h equal the space through which the body must pass to acquire the velocity u by the action of the force. Then

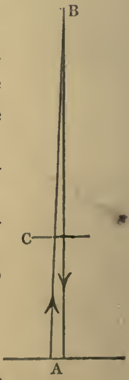
$$u^2 = 2fh$$

and for the space $h \pm s$ we have

$$\begin{aligned} v^2 &= 2f(h \pm s) \\ &= 2fh \pm 2fs \\ &= u^2 \pm 2fs \end{aligned}$$

The signs to be taken as in the last proposition.

13. Prop. *To find the time of flight or of ascent and descent of a body projected in an opposite direction to the action of the force.*



Let the body be projected from A with a velocity u. After ascending to some point B, it will return to A again. From the expression $v = u - ft$, we have for $t = 0$, or at time of projection $v = u$; at B, the highest point, $v = 0$, or $t = \frac{u}{f}$ = time of ascent; when $t = \frac{2u}{f}$, we have $v = u - f\left(\frac{2u}{f}\right) = -u$; or the velocity *acquired* in descending during any time is equal to the velocity *lost* in the same time in ascending.

Also, from $v^2 = u^2 - 2fs$, we have $s = \frac{u^2 - v^2}{2f}$; or s is the same when v is the same; if $AC = s$, we have the velocity at C the same when the body is ascending as when it comes to the same point again in descending, and the time in passing from C to B in the ascent is the same as in falling from B to C in the descent.

Consequently the whole time of flight from leaving A to coming to it again is $t = \frac{2u}{f}$.

If $v^2 > u^2$, s is negative, and must be measured below A.

EXAMPLES ON THE DIRECT ACTION OF GRAVITY AS A CONSTANT ACCELERATING FORCE.

Ex. 1. Find the velocity a stone will acquire in falling during four seconds by the action of gravity near the earth's surface.

In the expression $v=gt$, we have here $t=4$, $g=32{\cdot}2$,

$$\therefore v=4\times 32{\cdot}2=128{\cdot}8$$

or the stone has acquired a velocity of 128·8 feet per second.

Ex. 2. Find the space the stone in the last question had fallen through in three seconds.

Using the expression $s=\frac{1}{2}gt^2$, we have

$$\begin{aligned}\text{space required}&=\tfrac{1}{2}\times 32{\cdot}2\times 3^2\\&=144{\cdot}9\text{ feet}\end{aligned}$$

Ex. 3. Find the velocity the same stone had acquired when it had fallen through a space of 150 feet.

The expression $v^2=2gs$ gives us

$$\begin{aligned}v^2&=2\times 32{\cdot}2\times 150\\&=9660\end{aligned}$$

or $v=98{\cdot}3$ feet per second nearly, which is rather more than the velocity which would be acquired in three seconds.

Ex. 4. A heavy body is projected directly downwards with a velocity of 100 feet per second; what is its velocity at the end of five seconds?

The formula $v=u+gt$ gives

$$v=261\text{ feet per second.}$$

Ex. 5. A heavy body is projected directly upwards with a velocity of 100 feet per second; find its velocity at the end of five seconds.

We have now the expression $v=u-gt$,

$$\begin{aligned}\text{and here } v &= 100 - 32{\cdot}2 \times 5\\ &= -61\end{aligned}$$

The negative sign shews that the body is coming down again, and the velocity is 61 feet per second.

Ex. 6. Find how long the body in the last question continued to ascend.

At the highest point the velocity$=0$,

$$\therefore 0 = 100 - 32{\cdot}2 \times t$$

$$\text{or, } t = \frac{100}{32{\cdot}2} = 3{\cdot}1 \text{ seconds.}$$

Ex. 7. To find the height to which the body in Ex. 5 ascended.

The formula for this case is $v^2 = u^2 - 2gs$, and at the highest point $v=0 \;\; \therefore s = \frac{u^2}{2g}$,

$$\begin{aligned}\text{or, space ascended} &= \frac{100^2}{2 \times 32{\cdot}2}\\ &= 155{\cdot}2 \text{ feet.}\end{aligned}$$

Ex. 8. A body is projected vertically upwards, and returns to the same point in ten seconds; shew that the velocity of projection was 161 feet per second.

Ex. 9. Shew that when a body falls from rest by the action of a uniform accelerating force, the spaces described in successive equal intervals of time are as the series of odd numbers, 1, 3, 5, 7, 9, &c.

Ex. 10. A stone being let fall into a well, it is heard to strike the water in T seconds; required the depth of the well.

Let s be the depth of the well, m the velocity of sound$=1100$ feet per second, nearly.

Then T=time of the stone falling+the time of sound returning.

$$=\sqrt{\frac{2s}{g}}+\frac{s}{m}$$

$$\text{or, } s+\sqrt{s}\cdot\frac{m\sqrt{2}}{\sqrt{g}}=mT$$

a quadratic equation in $\sqrt{s}$; whence

$$\sqrt{s}=\pm\sqrt{Tm+\frac{m^2}{2g}}-\frac{m}{\sqrt{2g}}$$

which gives s, and the upper sign only is admissible.

Ex. 11. A body is thrown vertically upwards with a velocity u; find the time of its being at a given height h.

Let t be the time required,

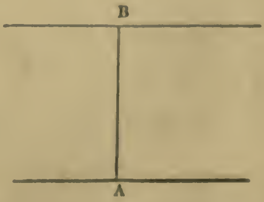
$$h=ut-\tfrac{1}{2}gt^2$$

whence
$$t^2-t\frac{2u}{g}+\frac{2h}{g}=0$$

$$\text{or, } t=\frac{u\pm\sqrt{u^2-2gh}}{g}$$

The lower sign gives the time as the body ascends, and the upper sign the time as it comes down again, when at the height h. At the highest point there will be only one value of t, and $\therefore \sqrt{u^2-2gh}=0$, or $h=\frac{u^2}{2g}$, as we should find by other modes.

Ex. 12. A body of given elasticity is projected upwards with a given velocity u to strike a horizontal plane, and in t seconds returns to the point of projection; required the distance of the plane from that point.



Let the body be projected from A with the given velocity u, and strike the horizontal plane at B with the velocity v.

Let the velocity of rebounding at B be v', and the modulus of elasticity be ϵ.

Let $AB = s$, the distance which is to be found.

Let t_1 be the time of ascending to B.

- t_2 - - - descending from B to A.

$$\text{Then } t_1 + t_2 = t \qquad (1)$$

$$\text{and } v = u - g t_1 \qquad v' = \epsilon v$$

$$= \epsilon (u - g t_1) \qquad (2)$$

$$s = u t_1 - \tfrac{1}{2} g t_1^2$$

$$= v' t_2 + \tfrac{1}{2} g t_2^2$$

Substituting for v' and t_2 their values from (2) and (1), we have a quadratic equation in t_1 which gives

$$t_1 = \frac{u + gt \pm \sqrt{(u+gt)^2 - \frac{4g}{1+\epsilon}(\epsilon ut + \frac{1}{2} g t^2)}}{2g}$$

Substituting this value of t_1 in the expression

$$s = u t_1 - \tfrac{1}{2} g t_1^2$$

we have s as required, the lower sign only being admissible.

Ex. 13. Shew that a body falling by the action of gravity acquires a velocity of 1000 feet per second in 31 seconds nearly.

Ex. 14. Shew that if a stone be thrown directly upwards with a velocity of 40 feet per second, it is returning down again after an interval of 1¼ seconds.

Ex. 15. Shew that the stone of the last example would ascend to a height of 24·8 feet.

Ex. 16. An imperfectly elastic ball falls upon a hard floor from a height h, shew that it will rebound to a height $\epsilon^2 h$.

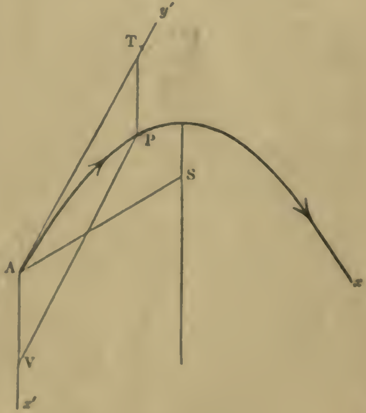
Ex. 17. A stone falling from a steeple passes through the last ¼ of the height in $\frac{1}{n}$th of a second; shew that the height of the steeple is $\frac{2g}{n^2}(7 + 4\sqrt{3})$.

CHAPTER IV.

ON PROJECTILES.

In the last chapter the motion of a body projected vertically upwards or downwards, under the action of gravity, was considered; the whole motion taking place in the vertical line through the point of projection. When the direction is any other than vertical, the path of the body is an arc of the curve called *the parabola*. By the second law of motion, gravity produces its full effect independent of the motion of projection: and we may consider the latter as compounded of a horizontal and vertical motion. The latter of these only can be affected by the action of gravity on the body.

14. Prop. *To determine the path of a body projected in a given direction, with a given velocity, under the action of gravity.*



Let v be the velocity of projection from the point A, in the direction ATy', and let t be the time in which the body would have described the space AT with the uniform velocity v, if gravity had not acted.

If TP be the space due to the action of gravity in the time t, P will be the actual place of the body.

Now, $AT = vt$ $\qquad PT = \frac{1}{2}gt^2$

eliminating t, $\qquad AT^2 = \dfrac{2v^2}{g}PT \qquad (1)$

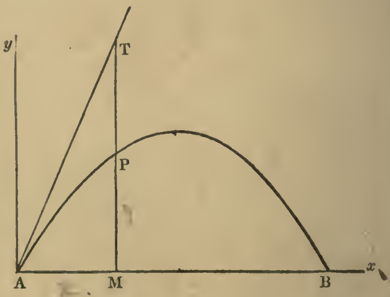
If we draw AVx' a vertical line, and taking $AV = PT$, complete the parallelogram $ATPV$, we have $AV = x'$, $PV = y'$, the oblique co-ordinates of the point P.

Let also h be the height from which the body must fall to acquire the velocity v, or $h = \frac{v^2}{2g}$, we have from (1)

$$y'^2 = 4hx'$$

which, as seen in treatises on conic sections, is the equation to a parabola whose axis is parallel to Ax', and therefore vertical, Ay' a tangent at the point A, and h the distance SA of the focus, and also of the directrix from A. With these data the parabola to represent the path of the body can be described.

15. PROP. *To find the equation to the path of a projectile when referred to axes of co-ordinates which are horizontal and vertical.*



Let v be the velocity of projection in the direction AT, which makes with Ax the *angle of elevation of the projectile* $TAx = \alpha$. Let APB be the path, and $AM = x$, $PM = y$, the co-ordinates of any point P. Let t be the time in which the body describes the arc AP; and let PM produced meet AT in T, we have

$$AT = vt \qquad TP = \tfrac{1}{2} g t^2$$

$$AM = x = vt \cos. \alpha \qquad PM = y = vt \sin. \alpha - \tfrac{1}{2} g t^2$$

Eliminating t, we have from these equations

$$y = x \tan. \alpha - x^2 \frac{g}{2v^2 \cos.^2 \alpha}$$

the equation required; or substituting, as in the last Prop., $h = \frac{v^2}{2g}$,

$$y = x \tan. \alpha - \frac{x^2}{4h \cos.^2 \alpha}$$

Definitions. *The horizontal range* of a projectile is the distance AB from the point of projection to the point where it strikes the horizontal plane in its descent. *The time of flight* is the time it takes in describing APB.

16. **Prop.** *To find the time of flight of a projectile on a horizontal plane.*

We have generally, as in the last proposition,

$$y = vt\sin.\alpha - \tfrac{1}{2}gt^2$$

and if we put $y = 0$, or $vt\sin.\alpha - \frac{1}{2}gt^2 = 0$, the result will apply to the points A and B only; and the values of t are

$$t = 0 \text{ at the point } A$$

$$t = \frac{2v\sin.\alpha}{g} \text{ at the point } B$$

which is therefore the time of flight required.

We should have arrived at this result by the same method as in Article 13, by putting for u, the vertical component of the velocity of projection, $v\sin.\alpha$.

17. **Prop.** *To find the range of a projectile on a horizontal plane.*

If we put $y = 0$ in the equation,

$$y = x\tan.\alpha - \frac{x^2.g}{2v^2.\cos.^2\alpha}$$

we have

$$0 = x\tan.\alpha - x^2\frac{g}{2v^2\cos.^2\alpha}$$

The result, as before, applies to the points A and B, and

$$x = 0 \text{ at the point } A$$

$$x = \frac{2v^2\sin.\alpha\cos.\alpha}{g} \text{ at the point } B$$

$$\text{or, } AB = \frac{v^2\sin.2\alpha}{g}$$

$$= 2h\sin.2\alpha$$

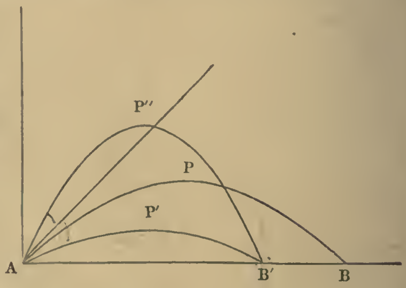
which is the horizonal range required.

This value of AB varies with the angle of elevation α, and is greatest when $\sin.2\alpha = 1$, or $\alpha = 45°$, v remaining the same.

Since $$\sin.\left(\frac{\pi}{2}+\theta\right) = \sin.\left(\frac{\pi}{2}-\theta\right)$$

$$\text{or, } \sin.2\left(45^\circ+\frac{\theta}{2}\right) = \sin.2\left(45^\circ-\frac{\theta}{2}\right)$$

if we put either $45^\circ+\frac{\theta}{2}$, or $45^\circ-\frac{\theta}{2}$, for α in the equation for AB, we have the same result; or, as in the annexed figure, if AB be the greatest range of a projectile when the angle of elevation is 45°, we shall have AB', a less horizontal range corresponding to either of the paths $AP'B'$, or $AP''B'$, when the elevation of one was as much above 45° as that of the other was below it.



We see that the horizontal range is the space which would be described with the uniform horizontal velocity $v\cos.\alpha$, in the time of flight $\frac{2v\sin.\alpha}{g}$.

18. Prop. *To find the greatest height which a projectile attains.*

The greatest altitude is evidently the value of y at the middle point of the path above a horizontal plane, or when the time is one-half of the time of flight, or $t = \frac{v\sin.\alpha}{g}$.

Putting this value of t in the expression

$$y = vt\sin.\alpha - \tfrac{1}{2}gt^2$$

$$\text{we have the greatest altitude} = \frac{v^2\sin.^2\alpha}{g} - \frac{1}{2}\frac{v^2\sin.^2\alpha}{g}$$

$$= \tfrac{1}{2}\cdot\frac{v^2\sin.^2\alpha}{g}$$

$$= h\sin.^2\alpha$$

Comparing this expression with $s = \frac{u^2}{2g}$, we see that the greatest altitude the projectile attains is that to which it would rise by the vertical component $v \sin. \alpha$ of the velocity of projection.

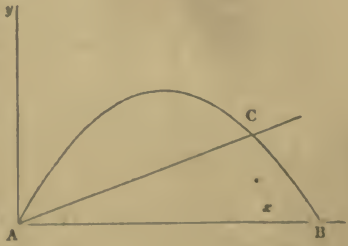
19. Prop. *To shew that the velocity at each point of the parabolic path of a projectile is that which would be acquired in falling directly from the directrix.*

In Article 14 it was shewn, that if h be the height due to the velocity of projection, it is also the distance of the point of projection from the directrix of the parabola described. Now, if any point in the parabola were taken for the point of projection, and a body were projected from it with the same velocity and direction which it has in the parabola, it would describe the same parabola; and therefore what holds for the point of projection holds also for all other points of the path.

20. Prop. *To find the point where a projectile will strike an inclined plane through the point of projection, and its distance, or the range on the inclined plane.*

Let $y = x \tan. \beta$ be the equation of the line AC, which is the intersection of the inclined plane with the vertical plane in which the body is projected.

Combining this with the equation of the path of the projectile in Article 15, namely,



$$y = x \tan. \alpha - \frac{x^2}{4h \cos.^2 \alpha}$$

we have the co-ordinates of the point C

$$x = 4h \frac{\cos. \alpha \, . \sin. (\alpha - \beta)}{\cos. \beta}$$

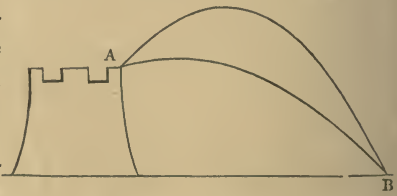
$$y = 4h \frac{\cos. \alpha \, . \sin. \beta \, . \sin. (\alpha - \beta)}{\cos.^2 \beta}$$

and the distance $AC = \sqrt{x^2 + y^2} = x \sec. \beta$

$$= 4h \frac{\cos. \alpha \, . \sin. (\alpha - \beta)}{\cos.^2 \beta}$$

EXAMPLES IN PROJECTILES.

Ex. 1. Two bodies are projected from the top of a tower with the same given velocity at different given angles of elevation, and they strike the horizontal plane at the same point. Required the height of the tower above the plane.



Let *A* be the point of projection, and *B* the point where the bodies strike the horizontal plane.

Since the velocity of projection is given, h, the height from which the bodies must fall to acquire it, is known. Let α and β be the given angles of elevation.

We have to find $-y$, the height of the tower, by eliminating x from the two equations,

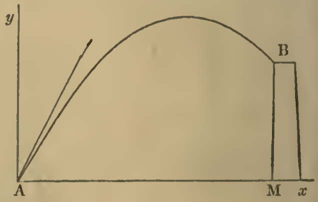
$$-y=x\tan.\alpha-\frac{x^2}{4h}(1+\tan.^2\alpha)$$

$$-y=x\tan.\beta-\frac{x^2}{4h}(1+\tan.^2\beta)$$

which give $y=\dfrac{4h}{(\tan.\alpha+\tan.\beta)\tan.(\alpha+\beta)}$

the height required.

Ex. 2. Find the angle of elevation at which a body must be projected with a given velocity in order to strike the summit of an object whose height and distance are given.



Let *A* be the point from which the body is projected to strike the point *B*, whose co-ordinates are $x=a$, $y=b$.

Putting these values in the equation to the path of the projectile, we find

$$\tan.\alpha=\frac{2h\pm\sqrt{4h(h-b)-a^2}}{a}$$

Ex. 3. Shew that the *latus rectum* of the parabolic orbit equals four times the space through which the body must fall to acquire the horizontal component of the velocity of projection.

Ex. 4. Shew that the time of flight on an inclined plane is given by the expression

$$\frac{4h\sin.(\alpha-\beta)}{v\cos.\beta}$$

Ex. 5. A body is projected with a velocity due to the height h, at an elevation α; shew that it ranges on a horizontal plane elevated a height H above the point of projection to a horizontal distance from that point

$$=h\sin.2\alpha\left\{1+\sqrt{1-\frac{H}{h\sin.^2\alpha}}\right\}$$

and that the time of flight is

$$\sqrt{\frac{2h}{g}}\sin.\alpha\left\{1+\sqrt{1-\frac{H}{h\sin.^2\alpha}}\right\}$$

CHAPTER V.

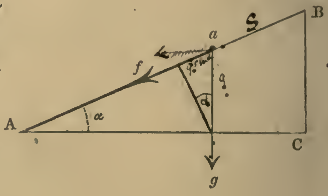
ON CONSTRAINED MOTION

WHEN a body acted on by any force is constrained to move in some particular manner, as, for instance, when a body falls down an inclined plane, or a curve, or swings as a pendulum, we call it a case of constrained motion.

21. PROP. *To determine the relations of the time, space, and velocity when a body falls by the action of gravity down an inclined plane.*

Let the body fall from the point B, down the inclined plane AB, whose inclination to the horizontal line AC is α.

Let a be its position at any time t from rest, $Ba = s$, v = velocity at a.



The force f which urges the body down the plane is the resolved part of gravity in that direction, or

$$f = g \sin. \alpha$$

which being a uniform force, we have only to put the value in the expressions found in Articles 9 and 10, for the required relations of s, t, and v.

The expressions $v = ft$, $s = \frac{1}{2}ft^2$, $v^2 = 2fs$, become respectively $v = g \sin. \alpha . t$, $s = \frac{1}{2}g \sin. \alpha . t^2$, $v^2 = 2g \sin. \alpha . s$, from which the whole circumstances of the motion can be determined.

22. PROP. *To shew that the velocity acquired in falling down an inclined plane is the same as would be acquired in falling down the perpendicular height directly.*

When the body arrives at A, the space described is AB; putting this value for s in the expression $v^2=2g\sin.\alpha.s$, we have

$$v^2=2g\,AB\sin.\alpha$$
$$=2g\,BC$$

the expression when the body falls directly through the height BC of the plane.

23. Prop. *To shew that the times down any inclined planes are proportional to the lengths of the planes, when the height is the same.*

Putting AB for s, and $\frac{BC}{AB}$ for $\sin.\alpha$ in the expression $s=\frac{1}{2}g\sin.\alpha.t^2$, we have

$$AB=\tfrac{1}{2}g\frac{BC}{AB}.t^2$$
$$\text{or, } t=AB\sqrt{\frac{2}{g.BC}}$$
$$\propto AB \text{ when } BC \text{ is constant.}$$

24. Prop. *To find the relations of the time, space, and velocity when a body is projected directly up or down an inclined plane.*

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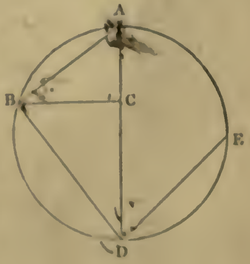
Putting for f its value $g\sin.\alpha$, on an inclined plane, in the expressions of Articles 11, 12, and 13, we have

$v=u\pm ft$ becoming on the inclined plane $v=u\pm gt\sin.\alpha$
$s=ut\pm\frac{1}{2}ft^2$ - - - - - $s=ut\pm\frac{1}{2}gt^2\sin.\alpha$
$v^2=u^2\pm 2fs$ - - - - - $v^2=u^2\pm 2gs\sin.\alpha$

25. Prop. *To shew that the times of a body falling down all the chords of a circle, in a vertical plane, drawn from the highest or to the lowest point, are the same.*



Let AD be the vertical diameter of the circle, AB a chord down which a body falls by the action of gravity. Draw BC horizontal. The accelerating force acting on the body is

$$f=g\frac{AC}{AB}$$

Putting $s=AB$ in the expression $s=\frac{1}{2}ft^2$, we have

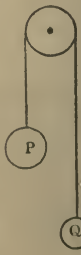
$$AB=\tfrac{1}{2}g\frac{AC}{AB}t^2$$

$$\text{or, } t^2=\frac{2}{g}\,\frac{AB^2}{AC}$$

$$=\frac{2}{g}AD \text{ . . . . since } AC : AB :: AB : AD$$

in a circle; or the time down any chord is the same as that down the diameter. The same relation evidently holds also for all chords drawn from the circle to the point D, as, for instance, DE or BD.

26. Prop. *Two bodies hang from the extremities of a cord passing over a pulley; to determine the motion.*



We shall first suppose that the weight of the pulley may be neglected.

Let P and Q be the weights of the bodies, of which P is the greatest; their masses are $\frac{P}{g}$ and $\frac{Q}{g}$ respectively. If the weights are equal, the bodies will balance, and no motion will ensue; but if one is heavier than the other, it will descend and the other rise through an equal space. The force which produces motion is the difference of the weights, which, being a moving force, equals the accelerating force multiplied by the mass moved.

Let f be the accelerating force, we have

$$P-Q=f\left(\frac{P+Q}{g}\right)$$

$$\text{or, } f=g\cdot\frac{P-Q}{P+Q}$$

This being a constant accelerating force, we shall have the circumstances of the motion by substituting its value in the expressions $v=ft$, $s=\frac{1}{2}ft^2$, $v^2=2fs$.

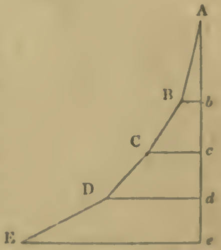
When we take into account the inertia of the pulley, we must add the equivalent mass acting at the cord to the other masses set in motion. Let I be this inertia or mass at the cord; we have

$$f=g\,\frac{P-Q}{P+Q+Ig}$$

This is the formala for Atwood's machine, referred to in the Article on the third law of motion. The force being uniform, and capable of being modified in any manner by changing the relative and absolute magnitudes of the weights, we are enabled by this machine to prove, in the most satisfactory way, by experiment, the formulæ for constant forces.

27. Prop. *When a body falls by the action of gravity down any arc of a smooth curve, the velocity at any point is that due to the vertical height fallen through.*

Let us first suppose that a body falls from A down a succession of smooth inclined planes AB, BC, CD, DE, &c., and loses no part of its acquired velocity in passing from one plane to the next. Draw $Abcde$, a vertical line, and Bb, Cc, Dd, Ee, &c., horizontal lines.



By Article 22, the body will have at B the same velocity as it would have acquired in falling directly through the vertical height Ab; and when it has passed without loss of velocity to the plane BC, it will have at B and all other points the velocity due to the vertical height fallen through. The same takes place on each of the other planes, and the velocities at the points C, D, E, &c. are those due to the vertical heights Ac, Ad, Ae, &c. respectively. When the number of planes is indefinitely increased, and the length of each indefinitely diminished, they form a continuous curve, which, when smooth, acts only perpendicular to the arc at each point, and therefore destroys by its reaction no part of the acquired velocity as the body passes from one point to another. Hence the velocity at all points of the curve is that due to the vertical height fallen through.

If a body be projected up or down a curve, the formulæ of Article 12 hold good; if we put u=velocity of projection, v= velocity after the body has passed through a *vertical height* h, we have

$$v^2=u^2\pm 2gh$$

The upper sign to be taken when the body is projected down, and the lower one when it is projected up the curve.

28. Prop. *When bodies fall by the action of gravity down any arcs of a circle in a vertical plane, the velocities at the lowest point are proportional to the lengths of the chords of the arcs.*

Let a body fall from the point B in the figure Article 25, down the arc BD; the velocity at D is that due to the vertical height fallen through, CD.

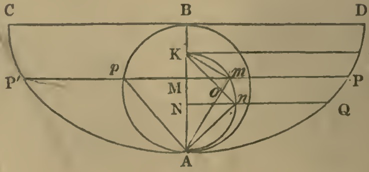
$$\begin{aligned}\text{or, vel.}^2&=2g\,CD\\&=2g\,\frac{BD^2}{AD}\\\text{or, vel.}&=BD\sqrt{\frac{2g}{AD}}\\&\propto BD\end{aligned}$$

When the arcs are small, the velocities are nearly proportional to the lengths of the arcs.

✓

29. Prop. *To find the time of a body falling down a cycloidal arc in a vertical plane with its base horizontal.*

The cycloid is a curve CAD described by a point in the circumference of a circle, whilst the circle rolls along the line CBD, called the base of the cycloid. The line AB, which



bisects the base CD perpendicularly, is called the axis of the cycloid, and is the diameter of the rolling circle when the point describing the cycloid comes to its central position A, which is called the vertex of the cycloid.

If a line PpP' be drawn parallel to the base CD, and Ap be the chord of the circle in its central position; it is shewn in mathematical treatises, that the arcs AP or AP' are equal to twice the chord Ap, and the tangent at P' is parallel to the chord Ap.

Let the body fall from the point L to any point P; and take Q a point near P. Draw the lines LK, PM, QN parallel to the base of the cycloid, meeting the axis in K, M, N respectively. Describe the semicircle $AnmK$ on AK, cutting PM in m, QN in n; and draw the chords Am, An, Km, Kn. Let o be the intersection of Kn and Am.

The velocity at the point $P = \sqrt{2g\,.\,KM}$, and if the arc PQ be indefinitely small, the velocity will be constant in it. Hence the time of describing $PQ = \dfrac{\text{space}}{\text{velocity}} = \dfrac{\text{arc } AP - \text{arc } AQ}{\sqrt{2gKM}}$.

$$= \frac{2 \times \text{chord of circle } ApB \text{ corresp}^{g} \text{ to } P - 2 \times \text{chord corresp}^{g} \text{ to } Q}{\sqrt{2gKM}}$$

$$= \frac{2\sqrt{AB\,.\,AM} - 2\sqrt{AB\,.\,AN}}{\sqrt{2gKM}}$$

$$= \sqrt{\frac{2AB}{g}} \left\{ \sqrt{\frac{AM}{KM}} - \sqrt{\frac{AN}{KM}} \right\}$$

$$= \sqrt{\frac{2AB}{g}} \left\{ \sqrt{\frac{AM\,.\,AK}{AK\,.\,KM}} - \sqrt{\frac{AN\,.\,AK}{AK\,.\,KM}} \right\}$$

$$= \sqrt{\frac{2AB}{g}} \left\{ \frac{Am}{Km} - \frac{An}{Km} \right\}$$

$$= \sqrt{\frac{2AB}{g}} \left\{ \frac{mo}{Km} \right\} \ldots \left\{ \begin{array}{l} \text{since ultimately } no \text{ becomes a cir-} \\ \quad \text{cular arc to center } A. \end{array} \right.$$

$$= \sqrt{\frac{2AB}{g}} \times \text{angle } mKo$$

The same holds for all other points between L and A; and the whole time of falling from L to A

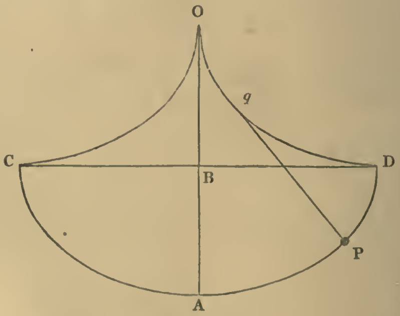
$$=\sqrt{\frac{2AB}{g}} \times \text{sum of all the small angles, as } mKn, \text{ in the right angle } LKA$$

$$=\frac{\pi}{2}\sqrt{\frac{2AB}{g}}$$

The body acquires in falling to A a velocity which will carry it through an equal arc on the opposite side of A, which it will describe in the same time; or if the time of the complete oscillation from L to an equal height on the opposite side of A be t, we have

$$t=\pi\sqrt{\frac{2AB}{g}}.$$

30. It is shewn in mathematical treatises, that if two half cycloids, CO, DO, equal to AD or AC, be constructed with their axes vertical, so that ABO is a straight line, and $BO=AB$, then the curve COD is the evolute of the cycloid CAD; or a curve such that a string of the length AO being fastened at O and wrapping on the arcs CO or DO, the extremity P of the string, when drawn tight, will be always in the cycloid CAD, the length of the arc $Oq+qP$ being always equal to OA.



If a body be suspended at P upon a cord fastened at O, wrapping and unwrapping on the arcs CO, OD, it will thus oscillate in a cycloidal arc.

Let $l=$ the length of the cord OA, the time of a complete oscillation is

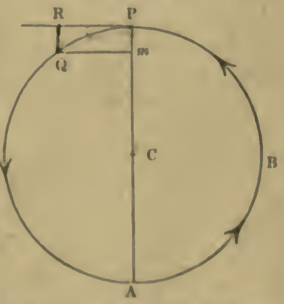
$$t=\pi\sqrt{\frac{l}{g}}$$

This is independent of the length of the arc of the cycloid described, and is consequently the same for all arcs. For this reason the cycloidal pendulum is called *isochronous*, and from this property arises the importance of the pendulum in instruments for measuring time; for small arcs near A the cord will not sensibly wrap upon the arcs of the evolute, and so a pendulum oscillating in *small* circular arcs has the property of isochronism. The clocks in astronomical observatories have their pendulums oscillating in very small arcs, which requires the mechanism of the clockwork to be very accurate, or the clock would be liable to stop going.

We learn from the expression $t=\pi\sqrt{\frac{l}{g}}$, that at the same place on the earth's surface, or when g is constant, $t \propto \sqrt{l}$; and at different points of the earth's surface if l is invariable, $t \propto \frac{1}{\sqrt{g}}$.

31. Prop. *A body at the extremity of an elastic cord, supposed without weight, describes a circle, with a uniform velocity, about a fixed point in the cord, as center; to find the tension in the cord.*

Let $PQAB$ be the circle in which the body moves, of which C is the center.



If the body were suddenly freed from constraint at any point P, it would, by the first law of motion, continue moving in the direction it had at P, and with its uniform velocity, or it would then move in the tangent PR, with that velocity. The effect of the tension in the cord is therefore to deflect the body from the tangent at each point in the circle, and is called a *centripetal* force. The tendency of the body to fly off by its inertia produces the force which balances this centripetal force, and is thence called the *centrifugal* force.

If PQ be an arc described in an indefinitely small time t, and PR the space which would have been described in the tangent, if the body were *free*, in the same time; then R and Q being supposed indefinitely near to P; RQ, by the second law of motion, becomes the space due to the action of the centripetal force.

Let PCA be a diameter of the circle, v the uniform velocity of the body.

Let $f=$ the centripetal accelerating force acting on the body; we have ultimately

$$RQ=Pm$$
$$=\tfrac{1}{2}ft^2$$
$$\text{and } PR=vt$$

But in the circle,

$$Qm^2=Pm\times mA$$
$$\text{or, } Pm=\frac{Qm^2}{mA}$$
$$=\frac{PR^2}{PA}, \text{ ultimately}$$
$$\therefore\ \tfrac{1}{2}ft^2=\frac{v^2t^2}{2PC}$$

or, if $r=$ the radius of circle,

$$f=\frac{v^2}{r}$$

If m be the mass of the body at P, the centrifugal moving force $=mf$

$$=\frac{mv^2}{r}$$

$=$the tension in the cord.

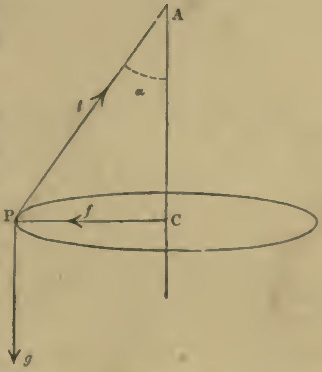
32. Definition. A body suspended by a cord which performs revolutions in a horizontal circle is called the *conical pendulum*.

33. Prop. *To determine the motion of a body in the conical pendulum.*

Let the body be fixed at the ex-mity P of the cord AP which is stened at A.

Let AC be a vertical line, $PC=r$ radius of the circle which the ly describes with the uniform ve-city v.

Let $l=$ the length of the cord P, and angle $PAC=\alpha$.



Since the body is in the same rcumstances at each point of the rcle, the forces acting upon it must balance each other. These rces are, the weight of the body acting downwards, the centri-gal pressure acting horizontally outwards from C, and the nsion in the cord AP.

Resolving horizontally and vertically, we have

$$t \sin.\alpha - \frac{m v^2}{r} = 0 \qquad (1)$$

$$t \cos.\alpha - m g = 0 \qquad (2)$$

From (2) we have the tension in the cord t

$$= \frac{m g}{\cos.\alpha}$$

$$= \frac{\text{the weight of the body}}{\cos.\alpha}$$

Eliminating t between (1) and (2), we have

$$v^2 = \frac{g r \sin.\alpha}{\cos.\alpha}$$

$$= g l \frac{\sin.^2\alpha}{\cos.\alpha}$$

The time of performing one revolution

$$= \frac{\text{circumference}}{\text{velocity}}$$

$$= \frac{2\pi r}{v}$$

$$= 2\pi\sqrt{\frac{l \cos.\alpha}{g}}$$

$$= 2\pi\sqrt{\frac{AC}{g}}$$

which depends on the vertical height AC, but not on the length of the cord.

34. Prop. *It is required to find the position of the rails in the curve of a railway, so that the resultant pressure of a carriage passing along the curve with a given velocity may be perpendicular to the line drawn from one rail to the other.*

Let APB represent in figure 1 the curve in the railway, and let the radius of the curve at P be $PC=r$. Let $v=$ the velocity of the carriage. If the reaction of the rails perpendicular to the line joining them as $a\,b$, figure 2, supplies the place of the tension in the cord in the last proposition, the carriage will pass smoothly and safely along the curve; for we shall then have the weight of the carriage acting at G, the center of gravity, figure 2, and the centrifugal pressure acting horizontally outwards giving a resultant pressure perpendicular to the rails which will be destroyed by their reaction.

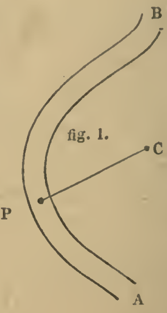


fig. 1.

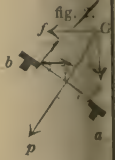


fig. 2.

If α be the angle which the resultant pressure in Gp makes with a vertical line = angle which $a\,b$ makes with a horizontal line, we have, from the last proposition,

$$v^2 = g\,r \tan.\,\alpha$$

$$\text{or, } \tan.\,\alpha = \frac{v^2}{g\,r}$$

which gives the inclination of the line $a\,b$ to the horizon as required.

EXAMPLES ON CHAPTER V.

Ex. 1. Two balls fall at the same instant from the common vertex down two inclined planes which meet the horizonta plane on which they rest at angles of 30° and 60°; shew tha the times of falling to the horizontal plane are in the ratio $\sqrt{3}:1$.

Ex. 2. To divide the length of a given inclined plane into three parts, so that the times of descent down them may be equal.

If the length of the plane be divided into nine equal parts, the body will descend down one in the first interval of time, down four in two intervals, and down the whole nine in the three equal intervals.

Ex. 3. P descending vertically draws Q up an inclined plane by means of a string passing over a pulley at the vertex; find the velocity acquired by P in describing a given space h.

Let P and Q be the weights of the bodies respectively, α the inclination of the plane to the horizon, h the space which P describes.

$$\text{the moving force} = P - Q\sin.\alpha$$

$$\text{the mass moved neglecting the pulley} = \frac{P+Q}{g}$$

$$\therefore \text{ the accelerating force} = g \cdot \frac{P - Q\sin.\alpha}{P+Q}$$

and from the general formula $v^2 = 2fs$ we find

$$\text{the velocity required} = \sqrt{2gh\frac{P - Q\sin.\alpha}{P+Q}}$$

Ex. 4. Twelve pounds weight is so distributed at the extremities of a cord passing over a pulley, that the more loaded end descends through seven feet in seven seconds. What is the weight at each end of the cord?

We find $P = 6\left(1 + \frac{1}{112 \cdot 7}\right)$, and $Q = 6\left(1 - \frac{1}{112 \cdot 7}\right)$ pounds.

Ex. 5. If three pendulums suspended in the same vertical plane have their lengths as the numbers 1, 4, and 9; shew that when they commence oscillating together, the first and second will be together again after four oscillations of the first; the first and third will be together again after three oscillations of the first; and the whole will be together again after twelve oscillations of the first.

Ex. 6. The length of the pendulum which beats seconds in London being 39·138 inches, shew that the force of gravity is represented by 32·19 feet nearly.

Ex. 7. If a body be projected obliquely upon an inclined plane, shew that its path is an arc of a parabola.

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THE END.

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